SUMMARY

OF

Some Aspects of Generalized de la Vallée Poussin Mean

THESIS PRESENTED FOR THE DEGREE OF

Doctor of Philosophy

IN

MATHEMATICS

OF THE

ALIGARH MUSLIM UNIVERSITY, ALIGARH

Under the Supervision of

Dr. Z. U. Ahmad, M. Sc, D. Phil, D. Sc.
READER

BY

Mohd Shuaib Siddiquie

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
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The thesis consists of eight chapters.

1. In chapter I, which is introductory, giving definitions and notations, we give some known results on the subject, which have direct interconnection with our investigations.

Let Σa_n be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of real constants, such that

$$P_n = p_o + p_1 + \ldots + p_n ,$$

and let us write

$$t_n = \frac{1}{P_n} \sum_{\nu=o}^{n} p_{n-\nu}\, s_\nu \quad (P_n \neq o).$$

Then the series Σa_n, or the sequence $\{s_n\}$ is said to be summable $|N,p_n|$, if $\{t_n\} \in$ BV, that is,

$$\Sigma |\, t_n - t_{n-1} \,| < \infty .$$

In the special case in which

$$p_n = A_n^{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, \Gamma(n+1)} , \ \alpha > -1 ,$$

the (N,p_n)- mean reduces to the familiar (C,α)-mean, and then $|N,p_n|$ is the same as $|C,\alpha|$.

Let $\lambda = \{\lambda_n\}$ be a monotone non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leq 1$ and $\lambda_1 = 1$, and let us write

$$V_n \equiv V_n(\lambda) = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v.$$

Then the series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is said to be summable (V,λ) to s, if

$$\lim_{n \to \infty} V_n = s.$$

The series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is said to be summable $|V,\lambda|$, if $\{V_n\} \in BV$, that is ,

$$\Sigma \mid V_{n+1} - V_n \mid < \infty.$$

Let $f(t)$ be a periodic function with period 2π and integrable L over $(-\pi,\pi)$. Then the Fourier series of $f(t)$ is given by

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$\equiv \sum_{n=0}^{\infty} A_n(t).$$

We write

$$\phi(t) = \frac{1}{2}\left\{f(x+t) + f(x-t)\right\}$$

and

$$\phi_1(t) = \frac{1}{t} \int_0^t \phi(t)\, dt.$$

We also write

$$\tau_n = \frac{1}{n} \sum_{v=1}^{n} v a_v, \quad \tau_n(x) = \frac{1}{n} \sum_{v=1}^{n} v\, A_v(x).$$

2. In chapter 2, we prove the following preliminary results for (V,λ)-mean.

Theorem 1. *If* $\lambda_n \neq 1$, *then the* (V,λ)-*method is regular, if, and only if* $\lambda_n \to \infty$, *as* $n \to \infty$.

The $(V,1)$-method is regular.

Theorem 2. *The* (V,λ)-*method is translative, whenever* $\lambda_n (\neq 1) \to \infty$, *as* $n \to \infty$. *The condition is also necessary if* λ_n *is an odd integer* $(\neq 1)$.

Combining Theorems 1 and 2 we get

Theorem 3. *If* $\lambda_n \neq 1$, *then the* (V,λ)-*method is translative if, and only if it is regular.*

Theorem 4. $(C,1)$ *is the only method which is both* (V,λ) *and* (N,p_n).

3. *Chapter 3 has been devoted to prove the following* inclusion theorems.

Theorem 5. If $\lambda_n/(n+1)$ is a sequence of bounded variation, then the summability $|V,\lambda|$ of $\{s_n\}$ implies the summability $|A|$ of $\{s_n^*\}$.[1]

Theorem 6. Let $p_n > 0$. Then the summability $|V,\lambda|$ of $\{s_n\}$ implies the summability $|N,p_n|$ of $\{s_n^*\}$.

4. In chapter 4 we prove the following theorem on $|V,\lambda|$-absolute convergence-factors.

Theorem 7. The necessary and sufficient condition that the series $\Sigma\, a_n^* \varepsilon_n$[2] should be absolutely convergent whenever $\Sigma\, a_n$ is summable $|V,\lambda|$, is that $\varepsilon_n = O(\frac{1}{\lambda_n})$.

This yields the following limitation theorem for $|V,\lambda|$-summability.

Theorem 8. If the series $\Sigma\, a_n$ is summable $|V,\lambda|$, then

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1. By s_n^* we mean :

$$s_n^* = \begin{cases} s_n - s_{n-\lambda_n}, & \text{when } \lambda_{n-1} = \lambda_n \\ s_n & \text{, when } \lambda_n = 1+\lambda_{n-1}. \end{cases}$$

2. By a_n^* we mean :

$$a_n^* = \begin{cases} a_n - a_{n-\lambda_n} & \text{when } \lambda_{n-1} = \lambda_n, \\ a_n & \text{when } \lambda_n = 1+\lambda_{n-1}. \end{cases}$$

<u>the series</u> $\Sigma\ |a_n|/\lambda_n$ <u>is convergent</u>.

5. In chapter 5, Theorem 8 have been used to prove the following result on the non-local character of $|V,\lambda|$-summability of a Fourier series at a point.

<u>Theorem 9. The summability</u> $|V,\lambda|$ <u>of the Fourier series of</u> $f(t)$, <u>at</u> $t = x$, <u>is not a local property of</u> $f(t)$, <u>that is, if</u> $0 < \alpha < \beta < 2\pi$, <u>there is a function</u> $f(t)$, <u>integrable</u> (L) <u>over the interval</u> $(x+\alpha,\ x+\beta)$ <u>and zero in the remainder of</u> $(x, 2\pi+x)$ <u>such that its Fourier series, at</u> $t = x$, <u>is not summable</u> $|V,\lambda|$.

6. In chapter 6, we prove

<u>Theorem</u> 10. <u>If</u> λ_n <u>is such that</u> $\left\{\frac{\lambda_{n+v}}{\lambda_n}\right\}$ <u>is bounded for</u> $v < n$,

$$\Sigma\ \frac{1}{(n+1)\ \lambda_n} < \infty\ ,$$

<u>and</u>

$$\Sigma\ \frac{|A_n(x)|}{\lambda_n} < \infty\ ,$$

<u>then the</u> $|V,\lambda|$-<u>summability of</u> $\Sigma\ A_n(t)$ <u>depends only on the behaviour of its generating function</u> $f(t)$ <u>in the immediate</u>

neighbourhood of the point $t = x$.

7. In chapter 7 the following theorems have been proved.

Theorem 11. If $\tau_n = O(\mu_n)$, as $n \to \infty$, where $\{\mu_n\}$ is a positive, non-decreasing sequence and if the sequence $\{e_n\}$ such that

$$\text{(i)} \quad \sum \frac{\mu_n}{\lambda_n} \mid e_n \mid < \infty ,$$

and

$$\text{(ii)} \quad \sum \mu_n \mid \Delta e_n \mid < \infty ,$$

then the series $\sum e_n a_n$ is summable $|V,\lambda|$.

Theorem 12. If $\phi_1(t) \in BV(o,\pi)$, and if the sequence $\{e_n\}$ is such that

$$\text{(i)} \quad \sum |e_n| / \lambda_n < \infty , \text{ and}$$

$$\text{(ii)} \quad \sum \mid \Delta e_n| < \infty ,$$

then the series $\sum e_n A_n(t)$, at $t = x$, is summable $|V,\lambda|$.

Theorem 13. If

$$\int_0^t u\,|d\ \phi_1(u)| = O(t),\ 0 \le t \le \pi ,$$

and if the sequence $\{\epsilon_n\}$ is such that

(i) $\Sigma (\log n/\lambda_n)\ |\epsilon_n| < \infty$,

and

(ii) $\Sigma (\log n)\ |\Delta \epsilon_n| < \infty$,

then the series $\Sigma\, \epsilon_n A_n(t)$, **at** $t = x$, **is summable** $|V,\lambda|$.

These theorems are the generalisations of the special cases of the results of Chow[1] and Prasad and Bhatt[2] respectively.

8. In the last chapter i.e. 8th chapter, we establish

Theorem 14. **If**

$$\sum_{v=1}^{n} \frac{|\tau_v|}{v} = O(\mu_n),$$

where $\{\mu_n\}$ is a positive non-decreasing sequence, and if the sequence $\{\epsilon_n\}$ is such that

(i) $\epsilon_n \mu_n = O(1),\ n\,\Delta \mu_n = O(\mu_n)$,

(ii) $\Sigma\, n\, \mu_n\ |\Delta^2 \epsilon_n| < \infty$,

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1. H.C.Chow : J.London Math.Soc.,29(1954),459-476,Theorem 2, for $\beta = \alpha = 1,\ p = o$.
2. B.N. Prasad and S.N.Bhatt : Duke Math.J. 24(1957),103-120, Theorems 5 and 7, for $\alpha = 1$.

then the series $\Sigma(n+1)^{-1}\lambda_n e_n a_n$ *is summable* $|V,\lambda|$.

Theorem 15. *If*

$$\Sigma\,|\tau_n| = O(n\,\mu_n)\,,$$

where $\{\mu_n\}$ *is a positive non-decreasing sequence, and if the sequence* $\{e_n\}$ *is such that* $\Sigma\,\frac{\mu_n}{\lambda_n}|e_n| < \infty$ *and*

(i) $\mu_n e_n = O(1)$, $n\,\Delta\mu_n = O(\mu_n)$,

(ii) $\Sigma\, n\,\mu_n\,|\,\Delta^2 e_n\,| < \infty$,

then the series $\Sigma\, e_n a_n$ *is summable* $|V,\lambda|$.

Theorem 16. *If the sequence* $\{e_n\}$ *is such that*

$\Sigma\,\frac{|e_n|}{\lambda_n} < \infty$, *and*

(i) $e_n = O(1)$ *and* (ii) $\Sigma\, n\,|\,\Delta^2 e_n| < \infty$,

then the series $\Sigma\, e_n A_n(x)$ *is summable* $|V,\lambda|$ *for almost all values of* x.

Theorem 17. *If* $F(x)$ *is even,* $F(x) \in L^2(-\pi,\pi)$,

$$\int_0^t |F(x)|^2\,dx = O(t)\,,$$

as $t \to +0$, and if $\{\epsilon_n\}$ satisfies the conditions

$$\text{(i)} \quad \log n \, \epsilon_n = O(1) ,$$

$$\text{(ii)} \quad \Sigma \, n \log n \, | \Delta^2 \epsilon_n | < \infty ,$$

then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\Sigma (n-1)^{-1} \lambda_n \epsilon_n A_n$ is summable $|V, \lambda|$.

Theorem 18. If $F(x)$ is even, $F(x) \in L(-\pi, \pi)$,

$$\int_0^t |F(x)| \, dx = o(t),$$

as $t \to +0$, and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem 18, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the propety that $\Sigma (n+1)^{-1} (\log n)^{-1/2} \lambda_n \epsilon_n A_n$ is summable $|V, \lambda|$.

Theorem 19. If $f(z) = \Sigma \, c_n z^n$ is a power series of the complex class L, such that

$$\int_0^t |f(e^{i\theta})| \, d\theta = o(|t|) ,$$

as $t \to +0$, and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem 18, then $\Sigma (n+1)^{-1} \lambda_n \epsilon_n c_n$ is summable $|V, \lambda|$.

These are the generalizations of the known results on $|C,1|$- summability and one for $|V,\lambda|$- summability.[1]

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1. T. Pati : *Math. Zeitschr.*, 78 (1962), 293-297; Theorem 1.

   C.T. Rajagopal : *Math. Zeitschr.*, 80(1963), 265-268, Theorems I, II and III.

   H.C.Chow : *J. London Math. Soc.* 16(1941), 215-220;

   L.Leindler : *Acta Sc. Math. Szeged*, 28(1967), 323-336; Theorem 1.

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Certificate

This is to certify that the contents of this thesis entitled " Some Aspects Of Generalized De La Vallée Poussin Mean " , is an original research work of Mr. Mohd. Shuaib Siddiqui, done under my supervision.

I further certify that the work of this thesis, either partly or fully has not been submitted to any other institution for the award of any other degree.

Z. U. Ahmad

(Z. U. Ahmad)
Supervisor

Counter-signed

HEAD
Department of Mathematics
Aligarh Muslim University
ALIGARH.

TO THE MEMORY OF MY FATHER

MOHD. ILYAS SIDDIQUI

Preface

The present thesis entitled, " Some Aspects Of Generalised De La Vallée Poussin Mean ", is the outcome of my researches that I have been pursuing since 1974 under the esteemed supervision of Dr. Z. U. Ahmad, M.Sc., D.Phil., D.Sc., Reader, Department of Mathematics, Aligarh Muslim University, Aligarh.

It has been my proud privilege to have accomplished my researches under the able supervision of Dr. Z.U.Ahmad who has made valuable contributions in the field of Absolute Summability. I have great pleasure in taking this opportunity of acknowledging my deep sense of gratitude and indebtedness to Dr. Ahmad for his inspiring guidance and constant encouragemant throughout the course of these researches.

The thesis consists of eight chapters. In chapter 1, which is introductory, giving the necessary definitions and notations which are used in the subsequent chapters, we mention known results in the subject, few of which have interconnections with our investigations. Chapter 2 concerns with some preliminary results characterising the generalized de la Vallée Poussin means. Chapter 3 deals with the inclusion relation of summability methods $|A|$ and $|N,p_n|$ with $|V,\lambda|$. In chapter 4, we prove $|V,\lambda|$ - Absolute convergence factor theorem which yields a limitation theorem for $|V,\lambda|$-summability and we use this theorem in chapter 5 to study the non-local character of $|V,\lambda|$-summability of Fourier series, while in chapter 6 we establish a localization theorem. The last two chapters, that is chapter 7 and 8, have been devoted to prove some results on $|V,\lambda|$-summability factor for general infinite series, which are applied to obtain results on $|V,\lambda|$-summability factors of power series and Fourier series.

Towards the end a comprehensive bibliography of various publications referred to in the body of the thesis has been given.

The contents of chapter 2 have been presented at the 66th session of the Indian Science Congress Association held at Hyderabad in January,1979. The matter of this and other chapters in the form of research papers have already been communicated to different international journals for favour of publication.

I wish to express my sincere appreciation to Prof. S. I. Husain, Head, Department of Mathematics,

A.M.U.,Aligarh, for his encouragement throughout my graduate studies and providing me the seminar and other facilities.

I also acknowledge to all my friends and colleagues whose valuable suggestions provided me a constant source of encouragement throughout my studies.

In the last but not the least my thanks go to Mr. Nafis Alvi for his excellent job of typing this thesis.

The 7th August 1979

M.S. Siddiquie

(MOHD.SHUAIB SIDDIQUIE)

Contents

Chapter I

INTRODUCTION

1.1. The use of infinite process in Analysis has its origin in the works of Carl Friedrich Gauss. His poineering efforts in formulating the idea of convergence of a power series to a function can be considered to be a major break-through in dealing with infinite processes. However, a rigorous formulation of convergence of general infinite series is due to Augustin-Louis Cauchy and was published in his famous treatise, course d'analyse de l'eclole polytechnique , Part I : Analyse Algebrique.

The success of Cauchy seems to be in asking the question how shall we assign a sum to an infinite series rather than asking what actually is the sum of an infinite series ? An infinite series which has a sum in Cauchy sense is said to be convergent. A series which is not convergent is called divergent.[1]

Neil's Henrik Abel[2] also made substantial contributions to the study of convergence and divergence of infinite series.

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1. For concept of convergence and divergence of infinite series, see Hobson [18a] and Knopp [21].
2. Abel [1].

The famous Abel's theorem is often used even to-day.

Cauchy's formulation of the concept of assigning a sum to a convergent infinite series led into a natural way to the question : ' Whether it was possible to develop a process which could assign a sum to even those series which were not convergent in Cauchy sense' ? Such a question was answered in affirmative and it was shown that it was possible to assign a sum to even divergent series in a variety of different senses which were more general than that of Cauchy convergence. These different processes of assigning sum to infinite series were termed as summability methods.[1] Just as the convergence of an infinite series led to the concept of summability, the absolute convergence led to the concept of absolute summability.[2]

The present thesis consists of the recent investigations of the auther into some aspects of the generalized de la Vallée Poussin mean, absolute summability by this mean and its applications to Fourier series.

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1. For a detailed study of summability methods, reference might be made to Hardy [18].
2. It was the pioneering works of Fekete ([16], and Kogbetliantz ([23],[24]) that this concept was introduced. For details see Zeller and Beekmann [49].

1.2. Most of the particular methods of summability are special cases of either one or the other of the two general methods :

(i) T- methods, and

(ii) ϕ- methods.

The T-methods are based upon the formation of a sequence of auxiliary means defined by sequence-to-sequence transformation :

(1.2.1) $\quad t_n = \sum_m c_{n,m}\, s_m, \qquad (n = 0,1,2,\ldots),$

$\{s_m\}$ being the m^{th} partial sum of a given series $\sum a_m$, and $c_{n,m}$ being the element of the n^{th} row and m^{th} column of the matrix $\| T \| = (c_{n,m})$, the matrix of summability.

The ϕ-methods are based upon the formation of a sequence-to-function transformation :

(1.2.2) $\quad t(x) = \sum_n \phi_n (x)\, s_n ,$

or the function-to-function transformation :

(1.2.3) $\quad t(x) = \int \phi(x,y)\, s(y)dy ,$

where x is a continuous parameter, $\phi_n(x)$ (or $\phi(x,y)$) is defined over an appropriate interval of x (or x and y).

A series Σa_n, or the sequence of its partial sums $\{s_n\}$, is said to be summable to a finite number s, by a T-method or a ϕ-method according as the sequence $\{t_n\}$, or the function $t(x)$, tends to s, as n tends to infinity or x tends to the appropriate limit depending upon the method.

A series Σa_n is said to be absolutely convergent if $\Sigma |a_n| < \infty$, i.e.,

$$(1.2.4) \qquad \Sigma \mid s_n - s_{n-1} \mid < \infty .$$

Thus the absolute convergence of Σa_n may be defined as the bounded variation of $\{s_n\}$. Symbolically, we write the relation (1.2.4) as

$$\{s_n\} \in BV^{1} .$$

A series Σa_n, or the sequence of its partial sums

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1. We use this notation throughout the thesis. Similarly by '$t(x) \in BV(h,k)$' we mean that the function $t(x)$ is of bounded variation in the interval (h,k). When the variable with respect to which bounded variation holds is emphasised, BV is followed by the corresponding letter as suffix, as in '$f(x,t) \in BV_t(h,w)$'.

$\{s_n\}$, is said to be absolutely summable, to the sum s, by a T-method, or a $\not\!p$-method according as $\{t_n\} \in BV$, or $t(x) \in BV(h,k)$, where (h,k) is the suitable interval of variation of the continuous variable x, and further $t_n \to s$, as $n \to \infty$; $t(x) \to s$, as x tends to a suitable limit.

1.3. The sequence-to-sequence transformation (1.2.1) is said to be conservative (K) (or absolutely conservative (AK)) if the convergence (or absolute convergence) of sequence $\{s_n\}$ implies that of the sequence $\{t_n\}$ in each case and is said to be regular (T) [or Absolutely regular (AT)], if further $\lim_{n \to \infty} s_n = s$ will imply $\lim_{n \to \infty} t_n = s$.[1]

Morley[2] has shown that a transformation may be AK without being K.

The necessary and sufficient conditions that the transformation (1.2.1) should be K, are :[3]

$$\text{(i)} \lim_{n \to \infty} c_{n,m} = \delta_m , \quad (m = 0,1,2,\ldots)$$

$$(1.3.1) \quad \text{(ii)} \lim_{n \to \infty} \sum_{m=0}^{\infty} c_{n,m} = \delta ,$$

$$\text{(iii)} \sum_{m=0}^{\infty} | c_{n,m} | \leq K^4.$$

1. For these concepts see Hardy [18] and Zeller and BeecKman[49].
2. Morley [32].
3. Hardy [18].
4. Throughout the Thesis K denotes an absolute constant, not necessarily the same at each occurence.

If, in addition $\delta_m = o$, for each m, and $\delta = 1$, then (1.3.1) gives the necess ry and sufficient conditions for the transformation to be T.

The necessary and sufficient conditions th t the transformation (1.2.1) should be AK, are :[1]

(1.3.2)

(i) $\sum_{m=o}^{\infty} c_{n,m} < \infty$, for each n,

(ii) $\sum_{n=o}^{\infty} \left| \sum_{m=\rho}^{\infty} (c_{n,m}-c_{n-1,m}) \right| \leq A \ (\rho = o,1,2,\ldots)$

(1.3.2) implies the existance of limits :

$$\lim_{n\to\infty} \sum_{m=o}^{\infty} c_{n,m} = \delta ,$$

$$\lim_{n\to\infty} c_{n,m} = \delta_m \quad (m = o,1,2,\ldots).$$

The transformation is absolutely re ular (T) if, in addition

$$\delta = 1 , \quad \delta_m = o \ (m = o,1,2,\ldots).$$

---

1. These conditions were first obtained by Mears [31], and the functional analytic proof of equivalent results were given later on by Knopp and Lorentz [22] and Sunouchi [43].

Similarly the necessary and sufficient conditions for the transformations (1.2.2) and (1.2.3) to be regular (T) [or absolutely regular (AT)] are also known.

A method of summability (absolute summability) is said to be ineffective[1] if it sums convergent (absolute convergent) series alone.

1.4. Given two methods of summability (or absolute summability), P and Q, we write $P \subseteq Q$, or $Q \supseteq P$ for '' Q includes P'' or '' P is included in Q '' to mean that every sequence summable by P is also summable by Q.

If $P \subseteq Q$ and $Q \subseteq P$, the two methods P and Q are said to be equivalent and we write $P \sim Q$.

If $P \subseteq Q$ and there exists a sequence which is summable Q but not P, then we write $P \subset Q$.

The theorems giving the results $P \subseteq Q$ or $P \subset Q$ are called '' Abelian theorems ''.

If $P \subset Q$, then the following question can be raised : ' Can there be sequences $\{\epsilon_n\}$ such that $\Sigma\, a_n \epsilon_n$ is

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1. Pati [35]

summable P whenever Σa_n is summable C' ?

The sequence like $\{e_n\}$ that are required to answer this question in affirmative are called summability factors (or absolute summability factors), and the corresponding results are called '' summability (absolute summability) factor theorems''.

The aim of the present thesis is to investigate the problems pertaining to Abelian theorems for absolute summability, absolute summability factor theorems and its applications to Fourier series, by Generalized de la Vallée Poussin Mean. We also discuss some preliminary results on Generalized de la Vallée Poussin mean and some other aspects of Fourier series, with respect to absolute summability by this mean.

In the sequel we give some special methods of summability with which we are directly concerned in this thesis.

1.5. Nörlund method.

In the special case in which

$$(1.5.1) \qquad c_{n,m} = \begin{cases} p_{n-m}/P_n \ , & m \leq n \\ o \ , & \text{otherwise} \ , \end{cases}$$

where $\{p_n\}$ is a sequence of constants, real or complex, such that

$$P_n = p_o + p_1 + \dots + p_n \neq o ,$$

the transformation (1.2.1) reduces to Nörlund mean[1] , or (N,p_n)-mean, of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$.

The series Σa_n, or the sequence $\{s_n\}$, is said to be summable (N,p_n), to sum s, if

$$\lim_{n \to \infty} t_n = s ,$$

and it is said to be absolutely summable (N,p_n), or summable $|N,p_n|$, if $\{t_n\} \in BV$.

In the special case in which

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma\alpha\,\Gamma(n+1)}, \quad \text{for } \alpha > -1,$$

1. Nörlund [33], substantially the same definition was given by G.F.Woronoi in the Proceedings of the 11th Congress of Russian naturalists and scientists (in Russian), St.Petersburg 1902, pp.60–61. An English translation of this work of Woronoi with ''remarks of the translator'' by J.D.Tamarkin is contained in Woronoi [48].

the corresponding Nörlund mean reduces to the familiar (C,α)-mean[1], s_n^α given by

$$s_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu , \quad A_n^\alpha = \binom{n+\alpha+1}{n}.$$

Necessary and sufficient conditions for the regularity of the Nörlund mean are[2]:

$$(1.5.2) \begin{cases} p_n = o(|P_n|), \quad \text{as} \quad n \to \infty , \\ \sum_{\nu=0}^{n} |p_\nu| = O(|P_n|), \text{ as } n \to \infty. \end{cases}$$

Generalized de la Vallée Poussin method .

Let $\lambda = \{\lambda_n\}$ be a monotone decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leq 1$, and $\lambda_1 = 1$. Then in the special case in which

$$c_{n,m} = \begin{cases} 1/\lambda_n , \text{ for } n - \lambda_n + 1 \leq m \leq n , \\ o \quad , \text{ otherwise,} \end{cases}$$

the transformation (1.2.1) reduces to V_n given by :

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1. See Hardy [18], § 5.13.

2. Hardy [18].

$$(1.5.3) \quad V_n = V_n(\lambda) = \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n+1}^{n} s_\nu \quad (n \geq 1),$$

which defines the sequence of generalized de la Vallée Poussin means, or (V,λ)-means, of the sequence $\{s_n\}$, generated by the sequence $\{\lambda_n\}$.

The series Σa_n, or the sequence $\{s_n\}$, is said to be summable by this mean, or simply summable (V,λ) to sum s, if

$$\lim_{n \to \infty} V_n = s.$$ [1]

The series Σa_n, or the sequence $\{s_n\}$, is said to be absolutely summable (V,λ), or summable $|V,\lambda|$ if $\{V_n\} \in BV$, that is to say,

$$(1.5.4) \quad \sum_{n=1}^{\infty} | V_{n+1} - V_n | < \infty.$$ [2]

In the special case in which $\lambda_n = n$, the corresponding (V,λ)-mean reduces to the familiar $(C,1)$-mean, and the corresponding methods are then $(C,1)$ and $|C,1|$.

The series Σa_n, or the sequence is said to be summable $|V,\lambda|_k$, $k \geq 1$, if

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1. Leindler [26].
2. Leindler [27].

$$(1.5.5) \qquad \sum_{n=1}^{\infty} \lambda_n^{k-1} \mid V_{n+1}(\lambda) - V_n(\lambda)\mid^k < \infty ,$$

For $\lambda_n = n$ it reduces to $|C,1|_k$, and for $k = 1$, it is the same as the summability $|V,\lambda|$.[1]

Abel summability .

In the special case in which

$$(1.5.6) \qquad \phi_n(x) = (1-x)\, x^n , \quad (o \leq x < 1)$$

the transformation (1.2.2) reduces to the Abel-transform, $f(x)$, given by :

$$(1.5.7) \qquad f(x) = (1-x) \sum_{n=o}^{\infty} s_n x^n = \sum_{n=o}^{\infty} a_n x^n .$$

The series Σa_n, or the sequence $\{s_n\}$, is said to be summable (A) to sum s, if the series on the right of (1.5.7) converges for $o \leq x < 1$, and if

$$\lim_{x \to 1-o} f(x) = s,$$[2]

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1. Varshney [45], see also Varshney [46].
2. Abel [2].

and is said to be absolutely summable (A), or summable $|A|$, if $f(x) \in BV[0,1)$, that is,

$$\int_0^1 |f'(x)|\, dx < \infty.$$[1]

For considerably detailed discussion concerning the consistency and equivalence of Nörlund methods reference may be made to Hardy[2], and of absolute Nörlund method to McFadden[3].

Absolute Cesàro summability was introduced by Fekete[4] for non-negative integral order and was studied by Kogbetliantz[5] in considerable details, who proved that

(i) $|C,\alpha| \subseteq |C,\beta|$, for every $\beta > \alpha > -1$, while

(ii) in general $|C,\alpha| \nsubseteq (C,\beta)$ if $\beta < \alpha$, and $|C,\alpha| \nsubseteq |C,\beta|$ for $\beta > \alpha$.

As regards to Abel's methods, analogous to Abel's[6]

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1. Prasad [39], Whittaker [47].
2. Hardy [18].
3. McFadden [30]. See also Zeller und Beeckmann [49].
4. Fekete [16].
5. Kogbetliantz [24].
6. Abel [1].

classical theorem, Whittaker[1] proved that $|C,o| \subseteq |A|$, Fekete[2] proved that $|C,\alpha| \subseteq |A|$, however large $\alpha > o$ may be, and also showed by means of negative example, that $|A| \nsubseteq (C,\alpha)$, and hence $|A| \nsubseteq |C,\alpha|$, however large $\alpha > o$ may be.

1.6. Absolute summability factors

Concerning the $|V,\lambda|$ - , and $|V,\lambda|_k$-summability factors of general infinite series the following results are known, which are the generalisations of the corresponding theorems for $|C,1|$- and $|C,1|_k$ summability.[3]

Theorem 1.[4] Let $\{\epsilon_n\}$ be a convex sequence such that $\Sigma \lambda_n^{-1} \epsilon_n < \infty$. If

$$\sum_{v=1}^{n} \frac{|s_v|}{\lambda_v} = O(\mu_n),$$

where $\mu_n = \sum_{v=1}^{n} \lambda_v^{-1}$, then $\Sigma a_n \epsilon_n$ is summable $|V,\lambda|$.

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1. Whittaker [47].
2. Fekete [17].
3. See, e.g. Pati [34], Mazhar [29], Singh [42] and Umar [44].
4. See Varshney [45], [46].

Theorem 2.[1] If $\{\epsilon_n\}$ is a convex sequence such that $\Sigma\, \lambda_n^{-1}\, \epsilon_n < \infty$, and

$$\sum_{v=1}^{n} \frac{|\, s_v\,|^k}{\lambda_v} = O(\mu_n\,),$$

where $\mu_n = \sum_{v=1}^{n} \lambda_v^{-1}$, then $\Sigma\, a_n\epsilon_n$ is summable $|V,\lambda|_k$.

Theorem 3.[1] If $\{\epsilon_n\}$ is a convex sequence such that $\Sigma\, \lambda_n^{-1}\, \epsilon_n < \infty$, and

$$\sum_{v=1}^{n} \frac{|s_v|}{\lambda_v} = O(\mu_n\gamma_n),$$

where $\mu_n = \sum_{v=1}^{n} \lambda_v^{-1}$ and $\{\gamma_n\}$ is a positive non-decreasing sequence such that

$$\lambda_n\mu_n\gamma_n\, \Delta\left(\frac{1}{\gamma_n} \right) = O(1), \text{ as } n \to \infty,$$

then $\Sigma\, a_n\epsilon_n/\gamma_n$ is summable $|V,\lambda|$.

Theorem 4.[1] If $\{\epsilon_n\}$ is a convex sequence such that $\Sigma\, \lambda_n^{-1}\, \epsilon_n < \infty$, and

$$\sum_{v=1}^{n} \frac{|s_v|^k}{\lambda_v} = O(\mu_n\gamma_n), \qquad (\, k \geq 1\,)$$

---

1. Varshney [45], [46].

where $\mu_n = \sum_{\nu=1}^{n} \lambda_\nu^{-1}$ and $\{\gamma_n\}$ is a positive non-decreasing sequence such that

$$\lambda_n \gamma_n \mu_n \Delta \left(\frac{1}{\gamma_n} \right) = O(1), \; n \to \infty$$

then $\sum a_n \epsilon_n/\gamma_n$ is summable $|V,\lambda|_k$.

In chapters 4, 7 and 8 , the author discusses some more problems on the $|V,\lambda|$-summability factors for general infinite series.

1.7. Absolute summability factors of Fourier series.

Let $f(t)$ be a periodic function with period 2π, and integrable in the sense of Lebesgue over $(-\pi,\pi)$. Then the Fourier series of $f(t)$ is given by :

$$f(t) \sim \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$\equiv \sum_{n=o}^{\infty} A_n(t),$$

where

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt, \; n=1,2,\ldots$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt, \; n=1,2,\ldots$$

We write

$$\phi(t) = \frac{1}{2}\left\{f(x+t) + f(x-t)\right\} .$$

In this context, the following results on $|V,\lambda|$ -and $|V,\lambda|_k$-summability factors of Fourier series are known, which are the generalizations of the corresponding results for $|C,1|$- and $|C,1|_k$-summability.[1]

Theorem 5.[2] If $\{\epsilon_n\}$ be a monotone convex sequence and the series $\Sigma\, \epsilon_n \lambda_n^{-1}$ converges, then the series $\Sigma\, \epsilon_n A_n(x)$ is summable $|V,\lambda|$ almost every-where.

Theorem 6.[2] If $f(x)$ belongs to the class H and if $\{\epsilon_n\}$ is monotone convex and satisfies the condition $\Sigma\, n\, \epsilon_n^2 \lambda_n^2 < \infty$, then the series $\Sigma\, A_n(x)\epsilon_n$ is summable $|V,\lambda|$ almost every-where.

Theorem 7.[2] Let $\epsilon(x)$ be a monotonically decreasing function and satisfies the condition :

$$(1.7.1) \qquad \sum_{n=1}^{\infty} \frac{\epsilon_n}{\lambda_n} < \infty .$$

---

1. See e.g. Chow [13], Hsiang [19] etc.
2. Leindler [27].

If

(1.7.2) $\phi(t) = o\,[\lambda^{-1}(\frac{1}{t})\,\epsilon(\frac{1}{t})\,]$

as $t \to +o$, then the series

(1.7.3) $\sum_{n=1}^{\infty} \epsilon(n)\, A_n(t)$,

is summable $|V,\lambda|$, at the point $t = x$.

If instead of (1.7.1) the condition

(1.7.4) $\sum_{n=4}^{\infty} \frac{\epsilon_n \log\log n}{\lambda_n} < \infty$

is satisfied, then the condition

(1.7.5) $\phi(t) = o(t\,(\log \frac{1}{t})^{-1}\,)$,

also suffices for the $|V,\lambda|$-summability of the series (1.7.3), at $t = x$.

Theorem 8.[1] If $\{\epsilon_n\}$ is a convex sequence such that $\sum \epsilon_n \lambda_n^{-1}$ is convergent, then the series $\sum \epsilon_n A_n(x)$ is summable $|V,\lambda|_k$, $k \geq 1$, almost every-where.

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1. See Varshney [46].

Theorem 9.[1] Let $\epsilon(x)$, $(x \geq 0)$ be a monotonic decreasing function such that

$$\sum_{n=1}^{\infty} \frac{\epsilon(n)}{\lambda_n} < \infty ,$$

and $\Phi(t) = \int_0^t |\phi(u)|du = O\left[\lambda^{-1}(\frac{1}{t})\ \epsilon\ (t)\ \right]$ holds, then the series

$\Sigma\ \epsilon(n)\ A_n(t)$, at $t = x$, is summable $|V,\lambda|_k$, $k \geq 1$.

Following theorem is also well-known for Fourier series :

Theorem 10.[2] Let $\lambda(x)$ $(x \geq 1)$ be a continuous function, linear between n and n+1 and let

$$\lambda(n) = \sqrt{(n\ \lambda_n)}\ .$$

If

$$\int_0^1 \frac{1}{t^2\ \lambda[\ \frac{1}{t}\ ]}\ \left[\ \int_0^{2\pi} (f(x+t)-f(x-t))^2 dx\right]^{1/2} dt < \infty, \tag{1.7.6}$$

---

1. See Varshney [46]

2. See Leindler [28].

then the Fourier series of $f(x)$ is $|V,\lambda|$-summable.

In chapters 7 and 8, the author discusses some more problems on $|v,\lambda|$-summability factors of Fourier series and power series. The problem of non-local character of $|V,\lambda|$-summability of Fourier series and its localization problem has also been discussed in chapters 5 and 6 respectively, while in chapter 2 some preliminary results on $|V,\lambda|$- mean have been established.

Chapter 2

ON THE METHOD OF SUMMATION BY GENERALIZED DE LA VALLÉE POUSSIN MEAN

2.1. Let Σa_n be a given infinite series with the sequence of partial sums, $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real constants, and let us write

$$(2.1.1) \qquad t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \qquad (P_n \neq 0)$$

where

$$P_n = p_0 + p_1 + \ldots + p_n.$$

The series Σa_n is said to be summable by Nörlund means, or summable (N,p_n), to the sum s, if $\lim_{n\to\infty} t_n = s$.[1] In the special case in which

$$p_n = A_n^{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,\Gamma(n+1)}, \ (\alpha > -1)$$

the (N,p_n)-method reduces to the familiar Cesàro-method (C,α) of order α $(\alpha > -1)$.[2]

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1. Nörlund [33], see also Woronoi [48].

2. Hardy [18], § 5.13.

Let $\lambda \equiv \{\lambda_n\}$ be a monotonic non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leq 1$, and $\lambda_1 = 1$. Then the sequence-to-sequence transformation

$$(2.1.2) \qquad V_n \equiv V_n(\lambda) = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v ,$$

defines the sequence $\{V_n\}$ of generalized de la Vallée Poussin means of the sequence $\{s_n\}$, generated by the sequence λ.[1]

The series Σa_n is said to be summable by the generalized de la Vallée Poussin means, or simply summable (V,λ), to the sum s, if

$$\lim_{n \to \infty} V_n = s.$$

When $\lambda_n = n$, (V,λ) is the same as $(C,1)$.

Put

$$(2.1.3) \qquad V'_n = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_{v-1}, \; s_0 = o .$$

Then, the method (V,λ) is called <u>translative if</u>

$$(2.1.4) \qquad \lim_{n \to \infty} (V_n - V'_n) = o ,$$

for all bounded sequences $\{s_n\}$.[2]

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1. Leindler [26].
2. Cf. Petersen [37].

2.2. Introduction. Recently Leindler[1] has defined the (V,λ)-mean and ordinary and absolute summability methods by this mean and their applications to Fourier series, orthogonal series and approximation of functions. He also gave the following formula :

$$(2.2.5) \qquad V_{n+1}-V_n = \frac{1}{\lambda_n\lambda_{n+1}} \sum_{\nu=n-\lambda_n}^{n+1} [(\lambda_{n+1}-\lambda_n)(\nu-n-1)+\lambda_n]a_\nu$$

In this chapter, we propose to prove some preliminary results concerning the (V,λ)-method, which seem to have not been discussed so far.

2.3. We prove the following theorems.

Theorem 1. If $\lambda_n \neq 1$, then the (V,λ)-method is regular if, and only if $\lambda_n \to \infty$, as $n \to \infty$.

The $(V,1)$-method is regular.

Theorem 2. The (V,λ)-method is translative whenever $\lambda_n (\neq 1) \to \infty$, as $n \to \infty$.

The condition is also necessary if λ_n is an odd integer $(\neq 1)$.

Combining Theorems 1 and 2 we get

Theorem 3. If $\lambda_n \neq 1$, then the (V,λ)-method is translative if, and only if it is regular.

Theorem 4. $(C,1)$ is the only method which is both (V,λ) and (N,p).

---

1. Leindler [25], [26], [27] and [28].

2.4. We need the following lemma for the proof of our theorems

Lemma 1.[1] <u>The necessary and sufficient conditions for the transformation</u> :

$$\sigma_n = \sum_{v=0}^{\infty} a_{nv} s_v$$

<u>to be regular, are</u> :

(i) $\lim\limits_{n \to \infty} \sum\limits_{v=0}^{\infty} a_{nv} = 1$,

(ii) $\lim\limits_{n \to \infty} a_{nv} = 0$, <u>for all</u> $v = 0,1,2,\ldots$,

(iii) $\sum\limits_{v=0}^{\infty} |a_{nv}| < K$, <u>for all</u> $n = 0,1,2,\ldots$,

<u>where</u> K <u>is a constant independent of</u> n.

2.5. <u>Proof of theorem</u> 1.

Since, by definition,

$$V_n = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v ,$$

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1. See Hardy [18].

(V,λ)-matrix is given by :

$$a_{n,v} = \frac{1}{\lambda_n}, \quad n - \lambda_n + 1 \leq v \leq n,$$

$$= 0, \text{ otherwise.}$$

Now, we see that

(i) $$\lim_{n \to \infty} \sum_{v=0}^{\infty} a_{n,v} = \lim_{n \to \infty} \sum_{v=n-\lambda_n+1}^{n} \frac{1}{\lambda_n} = 1,$$

(ii) $$\sum_{v=0}^{\infty} |a_{n,v}| = \sum_{v=n-\lambda_n+1}^{n} \frac{1}{\lambda_n} = 1,$$

and

(iii) $a_{nv} = \dfrac{1}{\lambda_n} \to 0$, as $n \to \infty$, for all $v = n-\lambda_n+1, \ldots,$ if and only if $\lambda_n \to \infty$, as $n \to \infty$, whenever $\lambda_n \neq 1$.

Therefore, in view of Lemma 1, the first part of Theorem 1 follows.

For the second part, we observe that, when $\lambda_n \equiv 1$, we get

$$V_n = s_n.$$

which gives

$$\lim_{n \to \infty} V_n = \lim_{n \to \infty} s_n.$$

This terminates the proof of Theorem 1.

2.6. Proof of Theorem 2.

By (2.1.2) and (2.1.3) , we have

(2.6.1) $$V_n - V'_n = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v - \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_{v-1}$$

$$= \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v - \frac{1}{\lambda_n} \sum_{v=n-\lambda_n}^{n-1} s_v$$

$$= \frac{s_n - s_{n-\lambda_n}}{\lambda_n}$$

Therefore, whenever $\lambda_n (\neq 1) \to \infty$, as $n \to \infty$, we have

$$\lim_{n \to \infty} (V_n - V'_n) = 0,$$

since, from (2.6.1), for bounded sequences,

$$V_n - V'_n \leq \frac{|s_n| + |s_{n-\lambda_n}|}{\lambda_n}$$

$$\leq \frac{K}{\lambda_n}.$$

This proves the first part of the theorem.

Next, let us consider the sequence $s_n = (-1)^n$, which is bounded. Then, from (2.6.1), we obtain

$$V_n - V'_n = \frac{(-1)^n}{\lambda_n} \left[1 - \frac{1}{(-1)^{\lambda_n}} \right]$$

$$= \begin{cases} \dfrac{2(-1)^n}{\lambda_n} & \text{, when } \lambda_n \text{ is an odd integer } (\neq 1); \\ 0 & \text{, when } \lambda_n \text{ is an even integer.} \end{cases}$$

Thus, whenever λ_n is an odd integer ($\neq 1$), is order that

$$\lim_{n \to \infty} (V_n - V_n') = 0,$$

we must have : $\lambda_n \to \infty$, as $n \to \infty$.

This completes the proof of Theorem 2.

2.7. Proof of Theorem.

Sufficiency clearly follows from the remarks of Section 2.1, so we have to prove necessity. Suppose then that (2.1.1) is of the form (2.1.2). Thus, for suitable p and λ,

(2.7.1) $$\frac{p_{n-v}}{P_n} = \frac{1}{\lambda_n}, \text{ for } n-\lambda_n+1 \leq v \leq n;\ n = 0,1,\ldots .$$

We first remark that, since (2.1.1) is unaltered if we replace p_n by $a\, p_n$, where a is any non-zero constant, and since $P_0=p_0 \neq 0$, there is no loss of generality in supposing that $p_0 = 1$.

Now, taking $v = n$ in particular, from (2.7.1), we have

$$\frac{p_0}{P_n} = \frac{1}{\lambda_n},$$

that is,

(2.7.2) $$P_n = \lambda_n.$$

Thus, from (2.7.1) and (2.7.2), we obtain

$$p_{n-v} = 1,$$

for $n-\lambda_n+1 \leq v \leq n$, and all n and all positive integers λ_n. This implies that

$$P_n = \lambda_n = n+1,$$

and with this value both the methods reduce to (C,1).

This completes the proof of Theorem 3.

Chapter 3

INCLUSION BETWEEN THE METHOD $|V,\lambda|$ AND THE METHODS $|A|$ AND $|N,p_n|$

3.1. Definitions and notations.

Let Σa_n be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, such that

$$P_n = p_o + p_1 + \dots + p_n.$$

Let us write

$$(3.1.1) \qquad t_n = \frac{1}{P_n} \sum_{v=o}^{n} p_{n-v}\, s_v \qquad (P_n \neq o).$$

The series Σa_n, or the sequence $\{s_n\}$ is said to be absolutely summable (N,p_n), or simply summable $|N,p_n|$, if

$$\{t_n\} \in BV.^1$$

In the special case in which

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1. Mears [31]

$$p_n = A_n^{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,\Gamma(n+1)} \qquad (\alpha > -1),$$

(3.1.1) reduces to the familiar (C,α)- mean[1] and then the summability $|N,p_n|$ is the same as absolute Cesàro summability $|C,\alpha|$.[2]

Let $\lambda \equiv \{\lambda_n\}$ be a monotone decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leq 1$ and $\lambda_1 = 1$. Then the sequence-to-sequence transformation :

$$(3.1.2) \qquad V_n \equiv V_n(\lambda) = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v$$

defines the sequence $\{V_n\}$, of generalised de la Vallée Poussin means, or (V,λ)-means, of the sequence $\{s_n\}$, generated by the sequence $\{\lambda_n\}$.

The series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is said to be summable $|V,\lambda|$, if $\{V_n\} \in BV$, that is,

$$\sum_n |V_{n+1} - V_n| < \infty.^3$$

---

1. Hardy [18], § 5.13.
2. Feket [16], Kogbetliantz [23], [24].
3. Leindler [27].

When $\lambda_n = n$, then $|V,\lambda|$ is the same as $|C,1|$.

The series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is said to be absolutely summable by Abel method (A), or summable $|A|$, if the series

$$(3.1.3) \qquad f(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n$$

converges for $0 \leq x < 1$ and if

$$f(x) \in BV\,(0,1).$$ [1]

We set

$$(3.1.4) \qquad \left(\sum_{n=0}^{\infty} p_n x^n\right)^{-1} = \sum_{n=0}^{\infty} c_n x^n, (|x| < 1;\ c_0 = 1)$$

Then, from (3.1.1) and (3.1.3), we get

$$(3.1.5) \qquad s_n = \sum_{v=0}^{n} c_{n-v}\, p_v\, t_v.$$

3.2. Introduction.

In this chapter we propose to prove a couple of inclusion theorems — one giving the relation $|V,\lambda| \subseteq |A|$ for a suitable λ_n and the other giving a necessary and sufficient condition for the relation $|V,\lambda| \subseteq |N,p_n|$.

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1. Whittaker [47], Prasad [39].

3.3. We establish the following theorems.

Theorem 1. If $\left\{\frac{\lambda_n}{n+1}\right\}$ is of bounded variation, then the summability $|V,\lambda|$ of the sequence $\{s_n\}$ implies the summability $|A|$ of the sequence $\{s_n^*\}$.[1]

Theorem 2. Let $p_n \geq 0$. Then the summability $|V,\lambda|$ of the sequence $\{s_n\}$ implies the summability $|N, p_n|$ of the sequence $\{s_n^*\}$, if and only if $\lambda_n = O(p_n)$.

3.4. We need the following lemmas for the proof of our theorems.

Lemma 1.[2] Let $\alpha_0 \neq 0$, and the power series $\sum_{n=0}^{\infty} \alpha_n x^n$ be convergent for $0 < x \leq \rho$ $(\leq \infty)$. If, uniformly for

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1. Throughout, by s_n^* we mean :

$$s_n^* = \begin{cases} s_n - s_{n-\lambda_n}, & \text{when } \lambda_{n-1} = \lambda_n \\ s_n, & \text{when } \lambda_n = \lambda_{n-1}+1, \end{cases}$$

2. Ahmad [5], Lemma 1; see also Ahmad [3], [4].

$$\left(\sum_{\nu=0}^{n} \alpha_\nu x^\nu\right) / \left(\sum_{\nu=0}^{\infty} \alpha_\nu x^\nu\right) \in BV_x(o, \rho),$$

then

$$\left(\sum_{n=0}^{\infty} \beta_n x^n\right) / \left(\sum_{n=0}^{\infty} \alpha_n x^n\right) \in BV_x(o, \rho),$$

whenever $\{\beta_n / \alpha_n\} \in BV$.

Lemma 2.[1] *If* $\alpha_n > o$, *for* $n = o, 1, 2, \ldots,$ *and the power series* $\sum_{n=0}^{\infty} \alpha_n x^n$ *converges for* $o \leq x < \rho$ $(\rho \leq \infty)$, *then uniformly in* $n \geq o$,

$$\alpha_n(x) \equiv \left(\sum_{\nu=0}^{n} \alpha_\nu x^\nu\right) / \left(\sum_{\nu=0}^{\infty} \alpha_\nu x^\nu\right) \in BV_x(o, \rho).$$

Lemma 3.[2] *Let* $\{x_n\}$ *be a sequence of real numbers and let*

$$y_n = \sum_{k=0}^{\infty} a_{n,k}\, x_k.$$

Then, for any $\sum_{n=0}^{\infty} |\Delta x_n| < \infty$ *implies* $\sum_{n=0}^{\infty} |\Delta y_n| < \infty$, *it is necessary and sufficient that* $\sum_{k=1}^{\infty} a_{n,k}$ *converges for all* n *and*

---

1. Ahmad [4], Lemma 2, see also Ahmad [3], [5].
2. Sunouchi [43], Theorem 3, see also Sarre [31].

$$\sum_{n=o}^{\infty} \Big| \sum_{k=1}^{m} (a_{n+1,k}-a_{n,k}) \Big| < K, \quad (m = 1,2,\ldots),$$

where K is an absolute constant independent of m.

3.5. Proof of Theorem 1.

By definition, we have

$$V_n = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v ,$$

and

$$V_{n-1} = \frac{1}{\lambda_{n-1}} \sum_{v=n-\lambda_{n-1}}^{n-1} s_v ,$$

so that, for $n = 1,2,\ldots,$

$$(3.5.1) \quad \lambda_n V_n - \lambda_{n-1} V_{n-1} = s_n^* = \begin{cases} s_n - s_{n-\lambda_n}, & \text{for } \lambda_{n-1} = \lambda_n , \\ s_n & , \text{for } \lambda_{n-1} = \lambda_n - 1 \end{cases} .$$

Now, we see that

$$f(x) = (1-x) \sum_{n=o}^{\infty} s_n^* x^n$$

$$= (1-x) \sum_{n=o}^{\infty} (\lambda_n V_n - \lambda_{n-1} V_{n-1}) x^n$$

$$= (1-x)^2 \sum_{n=o}^{\infty} \lambda_n V_n x^n$$

$$\frac{\sum_{n=o}^{\infty} \lambda_n V_n x^n}{\sum_{n=o}^{\infty} \lambda_n x^n} \cdot \frac{\sum_{n=o}^{\infty} \lambda_n x^n}{\sum_{n=o}^{\infty} (n+1) x^n} .$$

$$= L_1(x) . L_2(x)$$

$$\in BV_x(o,1),$$

by hypothesis and L mmas 1 and 2, since then

$$L_r(x) \in BV_x(o,1), \quad \text{for } r = 1,2.$$

Hence the theorem.

3.6. Proof of Theorem 2.

Again, since by definition,

$$V_n = \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} s_k ,$$

by (3.5.1), we have, for $n \geq 1$,

$$(3.6.1) \quad s_n^* = \begin{cases} s_n - s_{n-\lambda_n} = \lambda_n V_n - \lambda_{n-1} V_n, \text{ when } \lambda_{n-1} = \lambda_n, \\ s_n \qquad\qquad = \lambda_n V_n - (\lambda_n - 1) V_{n-1}, \text{when } \lambda_{n-1} = \lambda_n - 1, \end{cases}$$

$$= \sum_{v=o}^{\infty} a_{nv} V_v, \text{ say,}$$

where

$$(3.6.2) \quad a_{n,v} = \begin{cases} \lambda_n, \quad \text{when } v = n, \\ \left.\begin{array}{l} -\lambda_n, \quad \text{for } \lambda_{n-1} = \lambda_n \\ -\lambda_{n+1}, \text{for } \lambda_{n-1} = \lambda_n - 1 \end{array}\right\} \text{ when } v = n-1, \\ 0 \qquad , \text{ otherwise.} \end{cases}$$

Also, by definition, we have

$$t_n = \frac{1}{P_n} \sum_{v=o}^{n} p_{n-v}\, s_v^*$$

$$= \frac{1}{P_n} \sum_{v=o}^{n} p_{n-v} \sum_{k=o}^{v} a_{vk} V_k$$

$$= \sum_{k=o}^{\infty} b_{nk} V_k, \text{ say,}$$

where

$$b_{nk} = \frac{1}{P_n} \sum_{v=k}^{n} p_{n-v}\, a_{vk}, \text{ for } v \leq n,$$

$$= 0, \text{ otherwise.}$$

Since

$$\sum_{k=1}^{\infty} b_{nk} = \frac{1}{P_n} \sum_{k=1}^{n} \sum_{v=k}^{n} p_{n-v}\, a_{vk}$$

$$= \frac{1}{P_n} \sum_{v=1}^{n} p_{n-v} \sum_{k=1}^{v} a_{vk}$$

$$= \frac{1}{P_n} \sum_{v=1}^{n} p_{n-v} (\lambda_v - \lambda_{v-1})$$

$$= 0 \text{ or } 1,$$

according as $\lambda_{v-1} = \lambda_v$ or $\lambda_{v-1} = \lambda_v - 1$, by Lemma 3, $\Sigma|\Delta V_k| < \infty$ implies $\Sigma|\Delta t_n| < \infty$ if, and only if

$$(3.6.3) \quad \sum_{n=o}^{\infty} \Big| \sum_{k=1}^{m} (b_{n+1,k} - b_{n,k}) \Big| \leq K, \; m = 1,2,\ldots .$$

In particular,

$$\Big| \sum_{k=1}^{m} (b_{m+1,k} - b_{m,k}) \Big| = O(1),$$

or

$$(3.6.4)\qquad |(b_{m+1,m}-b_{m,m})+(b_{m+1,m-1}-b_{m,m-1})| = o(1).$$

But, since

$$b_{mm} = \frac{p_o}{P_m}\lambda_m \; ; \; b_{m,m-1} = \frac{p_o}{P_m}\lambda_{m-1} - \frac{p_1}{P_m}\lambda_{m-1}$$

$$b_{m+1,m} = -\frac{p_o}{P_{m+1}}\lambda_m + \frac{p_1}{P_{m+1}}\lambda_m$$

$$b_{m+1,m-1} = -\frac{p_1}{P_{m+1}}\lambda_{m-1} + \frac{p_2}{P_{m+1}}\lambda_{m-1},$$

we have

$$(b_{m+1,m}-b_{m,m}) + (b_{m+1,m-1},\ b_{m,m-1})$$

$$= \left(-\frac{p_o}{P_{m+1}}\lambda_m + \frac{p_1}{P_{m+1}}\lambda_m - \frac{p_o}{P_m}\lambda_m\right) + \left(-\frac{p_1}{P_{m+1}}\lambda_{m-1} + \frac{p_2}{P_{m+1}}\lambda_{m-1} - \frac{p_o}{P_m}\lambda_{m-1} + \frac{p_1}{P_m}\lambda_{m-1}\right)$$

$$\begin{cases} \lambda_m\left[\dfrac{p_2-p_o}{P_{m+1}} + \dfrac{p_1-2p_o}{P_m}\right], \text{ when } \lambda_{m-1} = \lambda_m \\[2ex] \lambda_m\left[\dfrac{p_2-p_o}{P_{m+1}} + \dfrac{p_1-2p_o}{P_m}\right] + \left[\dfrac{p_1-p_2}{P_{m+1}} + \dfrac{p_o-p_1}{P_m}\right], \text{when } \lambda_{m-1}=\lambda_m-1 \end{cases}$$

Then (3.6.4) gives $\lambda_m = O(P_m)$.

This proves the necessity part.

For sufficiency, we see that, whenever $\lambda_m=O(P_m)$,

$$\sum_{n=o}^{\infty} \left|\sum_{k=1}^{m} (b_{n+1,k} - b_{n,k})\right| \leq \frac{K\,\lambda_m}{P_m} \leq K.$$

This terminates the proof of Theorem 2.

Chapter 4

ABSOLUTE CONVERGENCE FACTORS FOR SERIES SUMMABLE $|V,\lambda|$

4.1. Introduction.[1]

In this chapter as Theorem 1 we prove that the method (V,λ) is absolute convergence preserving, that is, an absolutely convergent series is also summable $|V,\lambda|$. Then the question arises as to what type of sequences of factors $\{\epsilon_n\}$ can be determined such that the series $\sum a_n\epsilon_n$ should be absolutely convergent whenever $\sum a_n$ is summable $|V,\lambda|$. We settle this question by proving Theorem 2, which generalizes an special case of the following result of Bosanquet[2] for $\rho = o$ and $k = 1$.

Theorem A. If $o \leq \rho \leq k$, necessary and sufficient conditions for $\sum a_n\epsilon_n$ to be summable $|C, \rho|$ whenever

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1. Throughout this chapter we use the same definitions and notations as in the preceding chapter. We also take $s_o = a_o$ throughout this chapter.

2. Bosanquet [11], Theorem 2.

$\Sigma\, a_n$ is summable $|C,\rho|$, are :

$$\text{(i)}\quad \epsilon_n = O(n^{\rho-k}), \qquad \text{(ii)}\quad \Delta^k \epsilon_n = O(n^{-k}).$$

If $\rho > k \geq o$, the conditions are the same as in the case $\rho = k$.

As a corollary to Theorem 2, we obtain Theorem 3 which generalizes an special case of the following result of Kogbetlianz[1] when $\rho = o$, $\alpha = 1$.

<u>Theorem</u> B. If $\Sigma\, a_n$ is summable $|C,\alpha|$, then $\Sigma\, a_n/n^{\alpha-\rho}$ is summable $|C,\rho|$, for $o \leq \rho \leq \alpha$.

4.2. We establish the following theorems.

<u>Theorem</u> 1. $|C,o| \subset |V,\lambda|$.

<u>Theorem</u> 2. <u>The necessary and sufficient condition that the series</u> $\Sigma\, a_n^* \epsilon_n$[2] <u>should be absolutely convergent whenever</u> $\Sigma\, a_n$ is summable $|V,\lambda|$, is that $\epsilon_n = O\left(\frac{1}{\lambda_n}\right)$.

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1. Kogbetliantz [23]
2. By a_n^* we mean :

$$a_n^* = \begin{cases} a_n, & \text{when } \lambda_{n-1} = \lambda_n, \\ 0\ , & \text{when } \lambda_n = 1 + \lambda_{n-1}. \end{cases}$$

We obtain the following interesting theorem as a corollary to the above theorem.

Theorem 3. If $\Sigma\, a_n$ *is summable* $|V,\lambda|$, *then* $\sum_{n=1}^{\infty} |a_n| / \lambda_n$ *converges.*

4.3. We need the following lemma for the proof of the Theorem 2.

Lemma 1.[1] *Let* $\{x_n\}$ *be a sequence of real numbers and*

$$(4.3.1) \qquad y_n = \sum_{k=o}^{\infty} a_{n,k}\; x_k.$$

Then the necessary and sufficient condition for $\Sigma\, |y_n| < \infty$, *whenever* $\Sigma\, |x_n| < \infty$, *is that*

$$\sup_k \sum_{n=o}^{\infty} |\, a_{nk} \,| < \infty .$$

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1. Knopp and Lorentz [22].

4.4. Proof of Theorem 1.

The proof is easy, but we give it here for completeness.

It is well-known that[1]

$$V_{n+1}-V_n = \frac{1}{\lambda_n\lambda_{n+1}} \sum_{v=n-\lambda_n+2}^{n+1} [(\lambda_{n+1}-\lambda_n)(v-n-1)+\lambda_n]a_v.$$

Thus, in order to prove the theorem, it is enough to show that $\Sigma |V_{n+1} - V_n| < \infty$, whenever $\Sigma |a_n| = \Sigma |\Delta s_n| < \infty$.

Let $\sum_n'$ be the summation over all n satisfying $\lambda_{n+1}=\lambda_n$; and $\sum_n''$ the summation over all n where $\lambda_{n+1} > \lambda_n$. Then, since the lower indices $n - \lambda_n+2$ in $\sum_n'$ below are strictly increasing, we have

$$\begin{aligned}
\sum_n{}' &= \sum_n{}' \frac{1}{\lambda_n} \Big| \sum_{v=n-\lambda_n+2}^{n+1} a_v \Big| \\
&\leq \sum_n{}' \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+2}^{n+1} |a_v| \\
&= \sum_{v=2}^{\infty} |a_v| \sum_{n=v-1}^{v+\lambda_v-2} \frac{1}{\lambda_n} \\
&\leq K \sum_{v=2}^{\infty} |a_v| \\
&\leq K < \infty,
\end{aligned}$$

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1. See Leindler [27], proof of Theorem 1.

by hypothesis; and for $\lambda_{n+1} > \lambda_n$,

$$\sum_n'' = \sum_n'' \frac{1}{\lambda_n \lambda_{n+1}} \left| \sum_{v=n-\lambda_n+2}^{n+1} [(\lambda_{n+1}-\lambda_n)(v-n-1) + \right.$$

$$\left. + \lambda_n] a_v \right|$$

$$= \sum_n'' \frac{1}{\lambda_n \lambda_{n+1}} \left| \sum_{v=n-\lambda_n+2}^{n+1} (\lambda_n - n - 1 + v) a_v \right|$$

$$\leq K \sum_n'' \frac{1}{\lambda_n^2} \sum_{v=n-\lambda_n+2}^{n+1} \lambda_v |a_v| \quad (\text{since } |\lambda_n - n + v| \leq \lambda_v)$$

$$+ K \sum_n' \frac{1}{\lambda_n^2} \sum_{v=n-\lambda_n+2}^{n+1} |a_v|$$

$$\leq K \sum_{v=2}^{\infty} \lambda_v |a_v| \sum_{n \geq v}'' \frac{1}{\lambda_n^2}$$

$$\leq K \sum_{v=2}^{\infty} \lambda_v |a_v| \sum_{n \geq v}'' \frac{1}{\lambda_n^2}$$

$$\leq K \sum_{v=2}^{\infty} \lambda_v |a_v| \frac{1}{\lambda_v}$$

$$= K \sum_{v=2}^{\infty} |a_v| \leq K < \infty ,$$

since $\sum''$ has only the indices n having the property $\lambda_{n+1} > \lambda_n$, which gives

$$\sum_{n \geq v}'' \frac{1}{\lambda_v} \leq \sum_{\mu=\lambda_v}^{\infty} \frac{1}{\mu^2} \leq \frac{K}{\lambda_v} .$$

This terminates the proof of Theorem 1.

4.5. Proof of Theorem 2.

By definition, we have

$$V_n = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v, \text{ and } V_{n-1} = \frac{1}{\lambda_{n-1}} \sum_{v=n-\lambda_{n-1}}^{n-1} s_v,$$

so that

$$(4.5.1) \quad \lambda_n V_n - \lambda_{n-1} V_{n-1} = s_n^* = \begin{cases} s_n - s_{n-\lambda_n}, & \text{when } \lambda_n = \lambda_{n-1} \\ s_n & , \text{ when } \lambda_n = \lambda_{n-1} + 1 \end{cases},$$

and

$$(4.5.2) \quad \lambda_n V_n - 2\lambda_{n-1} V_{n-1} + \lambda_{n-2} V_n = a_n^* = \begin{cases} a_n - a_{n-\lambda_n}, & \text{when } \lambda_n = \lambda_{n-1} \\ a_n & , \text{when } \lambda_n = \lambda_{n-1} + 1 \end{cases}.$$

Thus, we have

$$(4.5.3) \quad s_n^* = \begin{cases} s_n - s_{n-\lambda_n} = \lambda_n (V_n - V_{n-1}), & \text{when } \lambda_n = \lambda_{n-1} \\ s_n = \lambda_n V_n - (\lambda_n - 1) V_{n-1}, & \text{when } \lambda_n = \lambda_{n-1} + 1 \end{cases},$$

and

$$(4.5.4) \quad a_n^* = s_n^* - s_{n-1}^* = \begin{cases} a_n - a_{n-\lambda_n} = \lambda_n (V_n - 2V_{n-1} + V_{n-2}), \text{when } \lambda_n = \lambda_{n-1} \\ a_n = \lambda_n V_n - 2(\lambda_n - 1) V_{n-1} + (\lambda_n - 2) V_{n-2}, \\ \qquad \text{when } \lambda_n = \lambda_{n-1} + 1. \end{cases}$$

Now, we have

$$(4.5.5) \quad y_n = \epsilon_n a_n^* = \sum_{k=0}^{\infty} b_{n,k} V_k,$$

where $(b_{n,k})$ is the matrix defined by :

$$b_{n,k} = \lambda_n \epsilon_n, \quad \text{when } k = n,$$

$$= \begin{cases} -2\lambda_n\epsilon_n , & \lambda_n = \lambda_{n-1} \\ -2(\lambda_n-1)\epsilon_n, & \lambda_n = \lambda_{n-1}+1 \end{cases}, \text{ when } k = n-1,$$

$$= \begin{cases} \lambda_n\epsilon_n , & \lambda_n = \lambda_{n-1} \\ (\lambda_n-2)\epsilon_n, & \lambda_n = \lambda_{n-1}+1 \end{cases}, \quad \text{when } k = n-2,$$

$$= 0, \qquad \text{otherwise.}$$

Then, by Abel's transformation, we have

$$(4.5.6) \qquad y_n = \sum_{k=0}^{\infty} B_{n,k} \triangle V_k + V B_n ,$$

where

$$B_{n,k} = \sum_{v=0}^{k} b_{n,v} , \; B_n = \lim_{n \to \infty} \sum_{v=0}^{m} b_{n,v} ,$$

and

$$V = \lim_{m \to \infty} V_m .$$

Since

$$B_n = \sum_{v=0}^{\infty} b_{n,v} = b_{n,n} + b_{n,n-1} + b_{n,n-2} = 0,$$

whenever $\lambda_n = \lambda_{n-1}$ or $\lambda_n = \lambda_{n-1}+1$, applying Lemma 1 to (4.5.6), we see that the necessary and sufficient condition that $\Sigma\ |y_n| < \infty$, whenever $\Sigma\ |\triangle V_n| < \infty$, is

$$\sum_{n=0}^{\infty} |B_{n,k}| \leq K, \text{ for every } k = 0,1,2,\ldots,$$

$$= 2\lambda_{n-1}\, \epsilon_n \quad , \quad k = n-1$$

$$= \lambda_{n-2}\, \epsilon_n \quad , \quad k = n-2$$

$$= o, \qquad \text{otherwise} .$$

Then, by Abel's transformation, we have

$$(4.5.5) \qquad y_n = \sum_{k=o}^{\infty} B_{n,k} \, \triangle V_k + V B_n ,$$

where

$$B_{n,v} = \sum_{v=o}^{k} b_{n,v} ,$$

$$B_n = \lim_{m \to o} \sum_{v=o}^{m} b_{n,v}$$

and

$$V = \lim_{m \to \infty} V_m .$$

Therefore, applying Lemma 1 to (4.5.5) we see that the necessary and sufficient condition that $\Sigma\, |y_n| < \infty$, whenever $\Sigma\, |\triangle V_n| < \infty$, is

$$\sum_{n=o}^{\infty} | B_{n,k} | \leq K \quad , \quad \text{for every } k = o,1,2,\ldots$$

or

$$(4.5.6) \qquad \sum_{n=o}^{\infty} \Big| \sum_{v=o}^{k} b_{nv} \Big| \leq K \quad ,$$

But by (4.5.4), we get

or

$$(4.5.7) \qquad \sum_{n=o}^{\infty} \left| \sum_{v=o}^{k} b_{n,v} \right| = O(1).$$

In particular, we get

$$\left| \sum_{v=o}^{k} b_{k,v} \right| = O(1),$$

that is,

$$| \epsilon_k | \, | \phi(\lambda_k) | = O(1).$$

But, since $\phi(\lambda_k) = 0$, whenever $\lambda_k = \lambda_{k-1}$, or $\lambda_k = \lambda_{k-1}+1$, we must have $\epsilon_k = O(\frac{1}{\lambda_k})$. For, if $\epsilon_k \neq O(\frac{1}{\lambda_k})$, then

$$\sum_{n=o}^{\infty} \left| \sum_{v=o}^{k} b_{n,v} \right| \neq O(1).$$

Hence the condition is necessary.

Next, whenever $\epsilon_k = O\left(\frac{1}{\lambda_k} \right)$, we have

$$\sum_{n=o}^{\infty} \left| \sum_{v=o}^{k} b_{n,v} \right| = |\epsilon_k| \, | [\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}] |$$

$$\leq \Sigma \; |\epsilon_k| \lambda_k = O(1).$$

This proves the sufficiency of the condition, and thus terminates the proof of Theorem 2.

Chapter 5

ON THE NON-LOCAL CHARACTER OF THE $|V,\lambda|$-SUMMABILITY OF A FOURIER SERIES AT A POINT

5.1. Definitions and Notations.

Let $f(t)$ be a periodic function with period 2π and intigrable (L) over $(-\pi,\pi)$, and let its Fourier series be given by

$$(5.1.1)\qquad \frac{1}{2}a_o + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

$$= \frac{1}{2}a_o + \sum_{n=1}^{\infty}A_n(t).$$

We write

$$\phi(t) = \frac{1}{2}\left\{f(x+t) + f(x-t)\right\}.$$

We use the other definitions and notations as given in Chapter 3.

5.2. Introduction.

It is known[1] that the summability $|C,\alpha|$, $\alpha > 1$ of

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1. Bosanquet [10].

the Fourier series of a Lebesgue integrable function, at a point, depends only upon the behaviour of the generating function in the immediate neighbourhood of the point, and is thus a local property. Bosanquet and Kestelman[1] have proved.

<u>Theorem</u> A. The summability $|C,1|$ of a Fourier series at a point is not a local property.

The object of this chapter is to prove an analogous result for the summability $|V,\lambda|$, which yields Theorem A as an special case when $\lambda_n = n$.

5.3. We establish the following theorem.

<u>Theorem</u> 1. <u>The summability</u> $|V,\lambda|$ <u>of the Fourier series of</u> $f(t)$, <u>at</u> $t = x$, <u>is not a local property of</u> $f(t)$, <u>that is, if</u> $o < \alpha < \beta < 2\pi$, <u>there is a function</u> $f(t)$, <u>integrable</u> (L) <u>over the interval</u> $(x + \alpha, x + \beta)$ <u>and zero in the remainder of</u> $(x, 2\pi + x)$ <u>such that its Fourier series, at</u> $t = x$, <u>is not summable</u> $|V,\lambda|$.

5.4. The following lemmas are pertinent for the proof of our theorem.

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1. Bosanquet and Kestelman [12], see also Daniel [15].

Lemma 1.[1] *Suppose* $f_n(x)$ *to be measurable in* (a,b), *where* $b - a \leq \infty$, *for* $n = 1,2,\ldots$, *then a necessary and and sufficient condition that, for every function* $\phi(x)$ *integrable* (L) *over* (a,b), *the function* $f_n(x).\phi(x)$ *should be summable over* (a,b) *and*

$$\sum_{n=1}^{\infty} \left| \int_a^b f_n(x)\, \phi(x)\, dx \right| < \infty ,$$

is that

$$\sum_{n=1}^{\infty} | f_n(x) |$$

is essentially bounded[2] *in* (a,b).

Lemma 2.[3] *If* Σa_n *is summable* $|V,\lambda|$, *then* $\Sigma |a_n|/\lambda_n < \infty$.

5.5. *Proof of the theorem.*

Let $o < \alpha < \beta < 2\pi$, and let $f(t)$ be integrable (L) in $(x+\alpha,\ x+\beta)$ and zero in the remainder of $(x,\ x+2\pi)$. We

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1. See Bosauquet and Kestelman [12]. See also Daniel [15].
2. $F(x)$ is said to be essentially bounded in (a,b) if $|F(x)| < K$ for almost all x in (a,b).
3. Theorem 4, Chapter 4.

assume (if possible) that the Fourier series $_n(t)$, at $t = x$, is summable $|V,\lambda|$. Then, by Lemma 2,

$$\sum_{n=1}^{\infty} \frac{|A_n(x)|}{\lambda_n} < \infty ,$$

what is the same thing,

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| \int_{\alpha} \phi(t) \cos nt \, dt \right| < \infty .$$

Therefore, in virtue of Lemma 1, we have

$$(5.5.1) \qquad \sum \frac{|\cos nt|}{\lambda_n} < K ,$$

for almost all t in $(\alpha,)$.

On the other hand, for $o < t < 2\pi$ $(t \neq)$,

$$\sum_n \frac{|\cos nt|}{\lambda_n} \geq \sum_n \frac{\cos^2 nt}{\lambda_n}$$

$$= \frac{1}{2} \sum_n \frac{1 + \cos 2nt}{\lambda_n}$$

$$\geq \frac{1}{2} \sum \frac{1}{\lambda_n} - \frac{1}{2} \left| \sum_n \frac{\cos nt}{\lambda_n} \right|$$

$$= \infty ,$$

since $\quad \Sigma \dfrac{1}{\lambda_n} \geq \Sigma \dfrac{1}{n} = \infty ,$

by hypothesis. But this contradicts (5.5.1), and the theorem follows in the case when $\lambda_n = \lambda_{n-1} + 1$.

Next, when $\cos^* nt = \cos nt - \cos(n-\lambda_n)t$, we have

$$\Sigma \frac{|\cos^* nt|}{\lambda_n} = 2\,\Sigma \frac{|\sin \frac{\lambda_n}{2} t \sin (n-\frac{\lambda_n}{2})t \;|}{\lambda_n}$$

$$\geq 2\Sigma \sin^2 \frac{\lambda_n}{2} t \sin^2 (n-\frac{\lambda_n}{2})\; t$$

$$\geq \frac{1}{2}\,\Sigma \frac{1}{\lambda_n} - \frac{1}{2} \left|\, \Sigma \frac{[\cos (2n-\lambda_n)+\cos \lambda_n t]}{\lambda_n} \right|$$

$$- \frac{1}{2} \left|\, \Sigma \frac{\cos (2n-\lambda_n) \cos \lambda_n t}{\lambda_n}\right.$$

$$= \infty ,$$

as before. This, again contradicts (5.5.1).

This completes the proof of the theorem.

Chapter 6

AN ASPECT OF LOCAL PROPERTY OF |V,λ|– SUMMABILITY OF FOURIER SERIES

6.1. Definitions and notations.

Let the function $f(t)$ be integrable (L) over $(-\pi,\pi)$ and periodic with period 2π, and let its Fourier series be

$$(6.1.1) \qquad \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv A_n(t).$$

We write

$$\phi(t) = f(x+t) + f(x-t).$$

We use other definitions and notations as given in Chapter 3.

6.2. Introduction.

In the preceding chapter we prove that, for a suitable λ_n, the summability $|V,\lambda|$ of a Fourier series is not a local property and obtain the corresponding results of Bosanquet and Kestelman[1] for $|C,1|$-summability as an special case.

1. Bosanquet and Kestelman [12].

The reason for this failure may be that the Lebesgue integrability of $f(t)$ over $(-\pi, \pi)$, by itself, does not warrant any thing beyond the asymptotic order estimate, viz.,

$$A_n(t) = o(1),$$

as $n \to \infty$. It may, therefore, be interesting to investigate whether, by imposing some-what greater restriction upon the behaviour of $A_n(t)$, we can make the summability $|V, \lambda|$ of (6.1.1), at a point, depend only on an additional condition of a local character, that is, a condition defining the behaviour of $f(t)$ in the immediate neighbourhood of the point considered. The answer to a question of this character is due to Jurkat and Peyerimhoff[1] for $|C, \alpha|$-summability, for $-1 < \alpha \leq 1$.

<u>Theorem</u> A. If $\sum \dfrac{|A_n(x)|}{n^\alpha}$ is convergent then the summability $|C, \alpha|$ of a Fourier series can be insured by a local property, where $-1 < \alpha \leq 1$.

The case $\alpha = 1$ has also been proved independently by Bhatt.[2]

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1. Jurkat and Peyerimhoff [20].
2. Bhatt [8].

The object of this chapter is to study the condition under which the summability $|V,\lambda|$ of a Fourier series at a point can be insured by a local condition.

6.3. We prove the following theorem.

Theorem. If λ_n is such that $\left\{\frac{\lambda_{n+v}}{\lambda_n}\right\}$ is bounded for $v < n$,

$$\sum \frac{1}{(n+1)\lambda_n} < \infty ,$$

and

$$\sum \frac{| A_n(x) |}{\lambda_n} < \infty ,$$

then the $|V,\lambda|$-summability of $\sum A_n(t)$ depends only on the behaviour of its generating function $f(t)$ in the immediate neighbourhood of the point $t = x$.

6.4. We require the folllwing lemmas on which the proof of the theorem depends.

Lemma 1. If the series

$$\sum \frac{| s_n |}{\lambda_n}$$

is convergent, then the series $\sum a_n$ is summable $|V,\lambda|$.

Proof of Lemma 1.

By definition

$$V_n = \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} s_k$$

and then

$$F_n \equiv V_{n+1}-V_n = \frac{1}{\lambda_n\lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} [(\lambda_{n+1}-\lambda_n)(k-n-1)+\lambda_n]a_k.$$

Let $\sum_n'$ be the summation over all n satisfying $\lambda_{n+1}=\lambda_n$; and $\sum_n''$ the summation over all n where $\lambda_{n+1} > \lambda_n$.

Thus, when $\lambda_{n+1} = \lambda_n$, we have

$$\sum_n' |F_n| = \sum_n' \frac{1}{\lambda_n} \left| \sum_{k=n-\lambda_n+2}^{n+1} a_k \right|$$

$$\leq \sum_n' \frac{|s_{n+1}|}{\lambda_n} + \sum_n' \frac{|s_{n-\lambda_n+1}|}{\lambda_n}$$

$$\leq K < \infty,$$

by hypothesis; and when $\lambda_{n+1} > \lambda_n$, by Abel's transformation, we have

$$F_n = \frac{1}{\lambda_n\lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} (\lambda_n+k-n-1)\, a_k$$

$$= \frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n} \Delta(\lambda_n + k - n - 1) s_k$$

$$+ \frac{s_{n+1}}{\lambda_{n+1}} - \frac{s_{n-\lambda_n+1}}{\lambda_n \lambda_{n+1}}$$

$$= F_{n,1} + F_{n,2} - F_{n,3} \text{, say ,}$$

so that

$$|F_n| \leq |F_{n,1}| + |F_{n,2}| + |F_{n,3}| .$$

Now, since

$$\sum_n{}'' (|F_{n,2}| + |F_{n,3}|) \leq 2 \sum_n{}' \frac{|s_{n+1}|}{\lambda_{n+1}} \leq K < \infty ,$$

and

$$\sum_n{}'' |F_{n,1}| \leq \sum_n{}'' \frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n} |s_k|$$

$$\leq \sum_{k=1}^{\infty} |s_k| \sum_{n \geq k}{}'' \frac{1}{\lambda_n^2}$$

$$\leq K \sum_{k=1} \frac{|s_k|}{\lambda_k}$$

$$\leq K < \infty ,$$

by hypothesis and the fact that

$$\sum_{n \geq k}{}'' \frac{1}{\lambda_n^2} \leq \frac{K}{\lambda_k} ,$$

we have

$$\sum_{n}'' |F_n| < \infty .$$

This proves the lemma.

Lemma 2. Let

$$I_n = \int_0^{\eta} \phi(u) \frac{\sin \frac{u}{2}}{\sin^2 \frac{\eta}{2}} \sin (n + \tfrac{1}{2}) u \, du$$

$$+ \int_{\eta}^{\pi} \phi(u) \frac{\sin(n + \frac{1}{2})u}{\sin \frac{u}{2}} \, du$$

Then $$I_n = O(1)\left(\sum_{v=-\infty}^{0} + \sum_{v=1}^{n-1} + \sum_{v=n+1}^{n+m} + \sum_{v=n+m+1}^{\infty} \right) \frac{|A_v|}{(n-v)^2} ,$$

where $A_v = A_v(x)$.

Proof of Lemma 2.

The proof of this lemma is contained in Bhatt.[1] We reproduce it here for completeness.

Let us define Ω (u) as follows :

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1. Bhatt [9], p. 787.

$$(6.4.1)\quad \Omega(u) = \begin{cases} (\sin \frac{\eta}{2})^{-2} \sin \frac{u}{2} & (o \leq u \leq \eta) \\ (\sin \frac{u}{2})^{-1} & (\eta \leq u \leq \pi). \end{cases}$$

Then, for $o \leq u \leq \pi$, $\Omega(u)$ is of bounded variation and continuous, with $\Omega(+o) = o$.

Also $\Omega'(u)$ is bounded and $\Omega''(u)$ is integrable (L). Now, since $\Omega(u)$ is of bounded variation in (o,π), by a well-known result[1] we have,

$$A_{-v}(x) = A_v(x) = A_v,$$

$$I_n = \frac{1}{2} A_o \int_o^\pi \Omega(u) \sin (n+\tfrac{1}{2})u \, du$$

$$+ \sum_{v=1}^{\infty} A_v \int_o^\pi \Omega(u) \cos vu \sin (n+\tfrac{1}{2})u \, du$$

$$= \frac{1}{2} \sum_{v=-\infty}^{+\infty} A_v \int_o^\pi \Omega(u) \sin (n-v+\tfrac{1}{2})u \, du$$

$$= -\frac{1}{2} \sum_{v=-\infty}^{+\infty} A_v \left[\Omega(u) \frac{\cos(n-v+\frac{1}{2})u}{n-v+\frac{1}{2}}\right]_o^\pi$$

$$+ \frac{1}{2} \sum_{v=-\infty}^{+\infty} A_v \int_o^\pi \Omega'(u) \frac{\cos(n-v+\frac{1}{2})u}{n-v+\frac{1}{2}} \, du$$

---

1. Hobson [18a], p.567.

$$= \frac{1}{2} \sum{}' A_v \int_0^{\pi} \Omega'(u) \frac{\cos(n-v+\frac{1}{2})u}{n-v+\frac{1}{2}} du$$

$$+ O(|A_n|),$$

where $\sum'$ denotes summation extending over $-\infty < v \leq n-1$ and $n+1 \leq v < +\infty$.

Let

$$\mu = \min (| n-v|^{-1}, \eta).$$

Then we have

$$(6.4.2) \qquad I_n = \frac{1}{2} \sum{}' A_v (\int_0^{\mu} + \int_{\mu}^{\pi}) \Omega'(u) \times$$

$$\times \frac{\cos(n-v + \frac{1}{2})u}{n- v+ \frac{1}{2}} du + O(|A_n|)$$

$$= I_{n,1} + I_{n,2} + O(|A_n|), \text{ say.}$$

Evidently,

$$(6.4.3) \qquad I_{n,1} = O(1) \left(\sum{}' \frac{|A_v|}{(n-v)^2} \right),$$

and

$$I_{n,2} = \frac{1}{2} \sum{}' A_v \left[\Omega'(u) \frac{\sin(n-v+ \frac{1}{2})u}{(n-v+ \frac{1}{2})^2} \right]_{\mu,\eta+0}^{\eta-0,\pi}$$

$$-\frac{1}{2}\sum{}' A_v \int_{\mu}^{\pi} \Omega''(u) \frac{\sin(n-v+\frac{1}{2})\,u}{(n-v+\frac{1}{2})^2}\,du,$$

where the integration by parts is taken separately over the ranges $(\mu, \eta-0)$ and $(\eta+0, \pi)$.

Thus

$$(6.4.4) \qquad I_{n,2} = O\left(\sum{}' \frac{|A_v|}{(n-v)^2}\right).$$

Then the result follows from (6.4.2), (6.4.3) and (6.4.4).

6.5. Proof of Theorem 6.3.

We have

$$s_n(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(u) \frac{\sin(n+\frac{1}{2})\,u}{\sin\frac{u}{2}}\,du$$

$$= \frac{1}{2\pi}\left(\int_0^{\eta}\phi(u) \frac{\sin\frac{u}{2}}{\sin^2\frac{\eta}{2}} \sin(n+\tfrac{1}{2})u\,du + \int_{\eta}^{\pi}\phi(u)\frac{\sin(n+\frac{1}{2})\,u}{\sin\frac{u}{2}}\,du\right)$$

$$+\frac{1}{2\pi}\int_0^{\eta}\phi(u)\left\{1-\left(\frac{\sin\frac{u}{2}}{\sin\frac{\eta}{2}}\right)^2\right\}\frac{\sin(n+\frac{1}{2})u}{\sin\frac{u}{2}}\,du$$

$$(6.5.1) \qquad = \frac{1}{2\pi}\,|I_n + J_n|, \quad \text{(say)}.$$

Hence, to prove the theorem, it is sufficient to show that the sequences $\{I_n\}$ and $\{J_n\}$ are summable $|V,\lambda|$. By virtue of Lemma 1, the sequence $\{I_n\}$ will be summable $|V,\lambda|$ if we prove that

$$(6.5.2) \qquad \sum \frac{|I_n|}{\lambda_n} < \infty .$$

But, by Lemma 2, we have

$$I_n = O(1) \left(\sum_{v=-\infty}^{0} + \sum_{v=1}^{n-1} + \sum_{v=n+1}^{n+m} + \sum_{v=n+m+1}^{\infty} \right) \frac{|A_v|}{(n-v)^2} + O(|A_n|)$$

$$= O(K_{n,1} + K_{n,2} + K_{n,3} + K_{n,4} + |A_n|), \text{ say.}$$

Now, we write

$$\sum_{n=1}^{\infty} \frac{K_{n,1}}{\lambda_n} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \sum_{v=o}^{\infty} \frac{|A_v|}{(n+v)^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left\{ \sum_{v=o}^{n-1} + \sum_{v=n}^{\infty} \right\} \frac{|A_v|}{(n+v)^2}$$

$$= L_{1,1} + L_{1,2}, \text{ say,}$$

so that, we have

$$L_{1,1} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \sum_{v=1}^{n-1} \frac{|A_v|}{\lambda_v} \frac{\lambda_v}{(n+v)^2}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{v=1}^{n-1} \frac{|A_v|}{\lambda_v}$$

$$\leq K \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\leq K \quad ,$$

by hypothesis ; and

$$I_{1,2} = \sum_{v=1}^{\infty} |A_v| \sum_{n=1}^{v} \frac{1}{(n+v)^2 \lambda_n}$$

$$\leq \sum_{v=1}^{\infty} \frac{|A_v|}{v} \sum_{n=1}^{v} \frac{1}{n \lambda_n}$$

$$= \sum_{v=1}^{\infty} \frac{|A_v|}{\lambda_v} \frac{\lambda_v}{v}$$

$$\leq \sum_{v=1}^{\infty} \frac{|A_v|}{\lambda_v} \quad ,$$

$$\leq K < \infty \quad ,$$

by hypothesis.

Again, as $m \to \infty$,

$$\sum_{n=2}^{m} \frac{K_{n,2}}{\lambda_n} = \sum_{v=1}^{m-1} \frac{1}{v^2} \sum_{n=v+1}^{m} \frac{|A_{n-v}|}{\lambda_n}$$

$$\leq \sum_{v=1}^{m-1} v^{-2} \sum_{n=v-1}^{m} \frac{|\mu_{n-v}|}{\lambda_{n-v}}$$

$$\leq K < \infty ,$$

by hypothesis; and as $m \to \infty$,

$$\sum_{n=1}^{m} \frac{K_{n,4}}{\lambda_n} \leq K \sum_{n=1}^{m} \frac{1}{\lambda_n} \sum_{v=n+m+1}^{\infty} \frac{1}{(n-v)^2}$$

$$\leq K \left\{ \frac{1}{m+1} \sum_{n=1}^{m} \frac{1}{\lambda_n} \right\}$$

$$\leq K \sum_{n=1}^{m} \frac{1}{(n+1)\lambda_n} \leq K < \infty ,$$

by hypothesis.

Lastly, as $m \to \infty$,

$$\sum_{n=1}^{m} \frac{K_{n,3}}{\lambda_n} = \sum_{n=1}^{m} \frac{1}{\lambda_n} \sum_{v=1}^{m} \frac{|\mu_{n+v}|}{v^2}$$

$$= \sum_{v=1}^{m} \frac{1}{v^2} \sum_{n=1}^{m} \frac{|\mu_{n+v}|}{\lambda_n}$$

$$= \sum_{v=1}^{m} \frac{1}{v^2} \left[\sum_{n=1}^{v} + \sum_{n=v+1}^{m} \right] \frac{|\mu_{n+v}|}{\lambda_n}$$

$$= L_{3,1} + L_{3,2}, \text{ say}$$

so that, as $m \to \infty$,

$$L_{3,1} = \sum_{v=1}^{m} v^{-2} \sum_{n=1}^{v} \frac{|A_{n+v}|}{\lambda_n}$$

$$\leq K \sum_{v=1}^{m} \frac{1}{v^2} \sum_{n=1}^{v} \frac{1}{\lambda_n}$$

$$= K \sum_{n=1}^{m} \frac{1}{\lambda_n} \sum_{v=n}^{m} \frac{1}{v^2}$$

$$\leq K \sum_{n=1}^{m} \frac{1}{(n+1)\lambda_n} \leq K < \infty ,$$

by hypothesis; and as $m \to \infty$,

$$L_{3,2} = \sum_{v=1}^{m} \frac{1}{v^2} \sum_{n=v+1}^{m} \frac{|A_{n+v}|}{\lambda_n}$$

$$= \sum_{v=1}^{m} \frac{1}{v^2} \sum_{n=v+1}^{m} \frac{|A_{n+v}|}{\lambda_{n+v}} \frac{\lambda_{n+v}}{\lambda_n}$$

$$\leq K \sum_{v=1}^{m} \frac{1}{v^2} \sum_{n=v+1}^{m} \frac{|A_{n+v}|}{\lambda_{n+v}}$$

$$\leq K < \infty ,$$

by hypothesis, since $\dfrac{\lambda_{n+v}}{\lambda_n} \leq K$, for $v < n$.

Thus finally, we obtain the result (6.5.2). We observe that, for positive η , however small, but fixed the summability $|V,\lambda|$ of the sequence $\{J_n\}$ depends only upon the behaviour of the generating function $f(t)$ in the immediate neighbourhood $(x-\eta , x+\eta)$ of the point x. This terminates the proof of the theorem.

Chapter 7

ON $|V,\lambda|$— SUMMABILITY FACTORS OF A FOURIER SERIES

7.1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let us write

$$\tau_n = \frac{1}{n} \sum_{v=1}^{n} v\, a_v .$$

Then the series $\sum a_n$ is said to be summable $|\ ,1|$, if

$$\sum \frac{|\tau_n|}{n} < \infty .$$

Let $\lambda \equiv \{\lambda_n\}$ be a monotone non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leq 1$, and $\lambda_1 = 1$. Then $\lambda_n \leq n$.

Let

$$V_n \equiv V_n(\lambda) = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v .$$

Then the series $\sum a_n$, or the sequence $\{s_n\}$, is said to be summable $|V,\lambda|$, if the series

$$\sum_{n=1}^{\infty} | V_{n+1} - V_n | ,$$

is convergent. If $\lambda_n = n$, then $|V,\lambda|$ is the same as $|C,1|$.

For any sequence $\{\epsilon_n\}$, we write

$$\Delta \epsilon_n = \epsilon_n - \epsilon_{n+1}, \quad \Delta^2 \epsilon_n = \Delta(\Delta \epsilon_n).$$

A sequence $\{\epsilon_n\}$ is said to be convex[1] if

$$\Delta^2 \epsilon_n \geq 0, \quad n = 0,1,2,\ldots$$

7.2. Let $f(t)$ be a periodic function, with period 2π, and integrable in the sense of Lebesgue over $(-\pi,\pi)$. We assume, as we may, without any loss generality, that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\,dt = 0 ,$$

and

$$f(t) \sim \Sigma\,(a_n \cos nt + b_n \sin nt) = \Sigma\, A_n(t).$$

Throughout we write

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1. Zygmund [50], p. 93.

$$\phi(t) = \frac{1}{2} \left\{ f(x+t) + f(x-t) \right\} ,$$

$$\phi_1(t) = \frac{1}{t} \int_0^t \phi(u)\, du$$

$$\phi_0(t) = \phi(t)$$

$$\zeta_n^*(x) = \frac{1}{n} \sum_{v=1}^{n} v A_v(x).$$

7.3. Introduction.

In 1954 Chow[1] proved the following theorem.

Theorem A. The necessary and sufficient conditions that the $\sum a_n \epsilon_n$ should be summable $|C,1|$, whenever $\zeta_n = O(1)$, as $n \to \infty$, are

$$\text{(i)} \quad \sum \left| \frac{\epsilon_n}{n} \right| < \infty ,$$

$$\text{(ii)} \quad \sum \left| \Delta \epsilon_n \right| < \infty .$$

For absolute Cesàro summability factors of a Fourier-Lebesgue series we know the following.

---

1. Chow [14]. Theorem 2 : This is an special case of the above referred theorem of Chow when $\alpha = \ = 1$, $p = o$. This result has recently been generalized by Ahmad [2] (in this form) and by Ahmad and Khan [7] (in its general form) for absolute Nörlund summability.

Theorem B.[1] If $\{\mu_n\}$ is convex sequence such that the series $\Sigma\, \mu_n/n$ is convergent, and $\phi_1(t) \in BV(0,\pi)$, then the series $\Sigma\, \mu_n A_n(t)$, at $t = x$, is summable $|C,1|$.

Theorem C.[2] If $\{\mu_n\}$ is a convex sequence such that the series $\Sigma\, \mu_n/n$ is convergent, and

$$\int_0^t u\,|d\,\phi(t)| = O(t),\ 0 \leq t \leq \pi\ ,$$

then the series $\Sigma\, (\log \overline{n+1}\,)^{-1} \mu_n A_n(t)$, at $t = x$, is summable $|C,1|$.

As most of the works done on $|V,\lambda|$-summability are generalizations of the results on $|C,1|$-summability, here, we also generalize the above mentioned theorems for $|V,\lambda|$-summability, so as to obtain the above referred theorems as special cases when $\lambda_n = n$. Theorem 1 gives $|V,\lambda|$-summability factors for general infinite series, while Theorem 2 and Theorem 3 are deduced from Theorem 1 as particular cases.

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1. This is an special case of Theorem 5 of Prasad and Bhatt [40], for $\alpha = 1$. This has been generalized by Ahmad [2] and Ahmad and Khan [7].
2. This is an special case of Theorem 7 of Prasad and Bhatt [40], $\alpha = 1$. This has been generalized by Ahmad [2] and Ahmad and Khan [7].

7.4. We establish the following theorems.

Theorem 1. *Let* $\tau_n = O(\mu_n)$, *as* $n \to \infty$, *where* $\{\mu_n\}$ *is a positive non-decreasing sequence and if the sequence* $\{\epsilon_n\}$ *such that*

$$\text{(i)} \quad \sum \frac{\mu_n}{\lambda_n} \, | \epsilon_n | < \infty ,$$

and

$$\text{(ii)} \quad \sum \mu_n \, | \Delta \epsilon_n | < \infty ,$$

then the series $\sum \epsilon_n a_n$ *is summable* $|V,\lambda|$.

Theorem 2. *If* $\phi_1(t) \in BV\,(o,\pi)$, *and if the sequence* $\{\epsilon_n\}$ *is such that*

$$\text{(i)} \quad \sum \; |\epsilon_n| \, / \, \lambda_n \; < \; \infty ,$$

and

$$\text{(ii)} \quad \sum \; |\Delta \epsilon_n \,| \; < \; \infty ,$$

then the series $\sum \epsilon_n A_n(t)$, *at* $t = x$, *is summable* $|V,\lambda|$.

Theorem 3. *If*

$$\int_o^t u\,|d\,\phi_1(u)\,| = O(t), \; o \leq t \leq \pi ,$$

and if the sequence $\{\epsilon_n\}$ *is such that*

$$\text{(i)} \quad \Sigma \left(\frac{\log n}{\lambda_n} \right) |\epsilon_n| < \infty ,$$

<u>**and**</u>

$$\text{(ii)} \quad \Sigma \; (\log n) \; |\Delta \epsilon_n| < \infty ,$$

<u>then the series</u> $\Sigma \, \epsilon_n A_n(t)$, <u>at</u> $t = x$, <u>is summable</u> $|V, \lambda|$.

Since a Lebesgue indefinite integral is absolutely continuous, $\phi_1(t) \in BV$ in every range (δ, π), $\delta > o$. Thus an interesting consequence of Theorem 2 is that the summability $|V, \lambda|$ of the series $\Sigma \, \epsilon_n A_n(x)$ is a local property.

7.6. <u>Proof of Theorem</u> 1.

By definition,

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} k \, a_k \, .$$

Hence
$$n a_k = n \tau_n - n \tau_{n-1}$$

$$= \sum_{v=1}^{n} a_{nv} \, \tau_v , \quad \text{say} ,$$

where

$$a_{nv} = \begin{cases} n, \text{ for } v = n , \\ n-1, \text{for } v = n - 1. \end{cases}$$

Let V_n^* be the n^{th} generalized de la Vallée Poussin mean of $\Sigma\, \epsilon_n a_n$. Then, by definition,

$$V_n^* = \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} s_k^* ,$$

where

$$s_k^* = \sum_{v=0}^{k} \epsilon_v a_v \quad ;$$

so that, by the well-known formula,

$$\sum_{n=1}^{\infty} | V_{n+1}^* - V_n^* | = \sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} [(\lambda_{n+1}-\lambda_n) \times \right.$$

$$\left. \times (k-n-1) + \lambda_n] \epsilon_k a_k \right| .$$

Let $\sum_n'$ be the summation over all n satisfying $\lambda_{n+1} = \lambda_n$; and $\sum_n''$ the summation over all n where $\lambda_{n+1} > \lambda_n$. Then,

$$\Sigma' = \sum_n' \frac{1}{\lambda_{n+1}} \left| \sum_{k=n-\lambda_n+2}^{n+1} \frac{\epsilon_k a_k}{k} \right|$$

$$= \sum_n' \frac{1}{\lambda_{n+1}} \left| \sum_{k=n-\lambda_n+2}^{n+1} \frac{\epsilon_k}{k} \sum_{v=1}^{k} a_{kv} \left(\frac{\tau_v}{\mu_v} \right) \mu_v \right|$$

$$\leq K \sum_n' \frac{1}{\lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} \frac{|\epsilon_k|}{k} \sum_{v=1}^{k} \mu_v | a_{kv} |$$

(by hypothesis)

$$= K \sum_n' \frac{1}{\lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n} \frac{|\epsilon_k|}{k} (k\mu_k - (k-1)\mu_{k-1})$$

$$+ K \sum_n{}' \frac{1}{\lambda_{n+1}} \frac{|\epsilon_{n+1}|}{n+1} [(n+1)\mu_{n+1} - n\mu_n]$$

$$= K \left(\sum_{n}{}_1' + \sum_{n}{}_2' \right) , \text{ say.}$$

Now

$$(7.6.1) \qquad \sum_{n}{}_2' = \sum_n{}' \frac{1}{\lambda_{n+1}} \frac{|\epsilon_{n+1}|}{n+1} [(n+1)\mu_{n+1} - n\,\mu_n]$$

$$= \sum_{n=2}^{\infty} \frac{1}{\lambda_n} |\epsilon_{n+1}| \mu_{n+1} - \sum_{n=2}^{\infty} \frac{1}{\lambda_n} \frac{|\epsilon_{n+1}|}{n+1} n\,\mu_n$$

$$= \sum_{n=1}^{\infty} \mu_{n+1} \left[\frac{|\epsilon_{n+1}|}{\lambda_{n+1}} - \frac{|\epsilon_{n+2}|}{\lambda_{n+2}} \right]$$

$$- \sum_{n=2}^{\infty} \frac{|\epsilon_{n+2}|}{\lambda_{n+2}(n+2)} \mu_{n+1} + K$$

$$\leq \sum_{n=1}^{\infty} \mu_{n+1} \left| \Delta \frac{(\epsilon_{n+1})}{\lambda_{n+1}} \right| + \sum_{n=2}^{\infty} \mu_{n+2} \frac{|\epsilon_{n+2}|}{\lambda_{n+2}} + K$$

$$= \sum_{n=1}^{\infty} \mu_{n+1} \frac{|\Delta\epsilon_{n+1}|}{\lambda_{n+1}} + \sum_{n=1}^{\infty} \mu_{n+1}(\lambda_{n+2}-\lambda_{n+1}) \frac{|\epsilon_{n+1}|}{\lambda_{n+1}}$$

$$\leq \sum_{n=1}^{\infty} \mu_{n+1} |\Delta\epsilon_{n+1}| + \sum_{n=1} \mu_{n+1} \frac{|\epsilon_{n+1}|}{\lambda_{n+1}} + K$$

$$\leq K ,$$

by hypothesis; and

$$\Sigma_n{}' = \Sigma_n{}' \frac{1}{\lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n} \frac{|\epsilon_k|}{k} [k\mu_k-(k-1)\mu_{k-1}]$$

$$\leq \sum_{k=1}^{\infty} \frac{|\epsilon_k|}{k} [k\mu_k-(k-1)\mu_{k-1}] \sum_{n=k}^{n=k+\lambda_k-1} \frac{1}{\lambda_n}$$

$$\leq \sum_{k=1}^{\infty} \frac{|\epsilon_k|}{k} [k\mu_k -(k-1)\mu_{k-1}] ,$$

where

$$(7.6.2) \quad \sum_{k=1}^{\infty} \frac{|\epsilon_k|}{k} [k\mu_k -(k-1)\mu_{k-1}]$$

$$= \sum_{k=1}^{\infty} (|\epsilon_k| - |\epsilon_{k+1}|)\mu_k - \sum_{k=1}^{\infty} \frac{|\epsilon_{k+1}|}{k+1} \mu_k$$

$$\leq \sum_{k=1}^{\infty} \mu_k|\Delta\epsilon_k| + K \sum_{k=1}^{\infty} \mu_{k+1} \frac{|\epsilon_{k+1}|}{\lambda_{k+1}} + K$$

$$\leq K ,$$

by hypothesis.

Next, when $\lambda_{n+1} > \lambda_n$, we have

$$\Sigma_n{}'' = \Sigma_n{}'' \frac{1}{\lambda_n \lambda_{n+1}} \left| \sum_{k=n-\lambda_n+2}^{n+1} (\lambda_n-n-1+k) \frac{\epsilon_k}{k} k a_k \right|$$

$$\leq K \Sigma_n{}'' \frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} \lambda_k \frac{|\epsilon_k|}{k} \sum_{v=1}^{k} |a_{kv}| \mu_v$$

$$= K \Sigma_n{}'' \frac{1}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} \lambda_k \frac{|\epsilon_k|}{k} [k\mu_k-(k-1)\mu_k]$$

$$+ K \sum_n{}'' \frac{1}{\lambda_n} \frac{\lambda_{n+1}}{\lambda_{n+1}} \frac{|\epsilon_{n+1}|}{n+1} [(n+1)\mu_{n+1} - n\mu_n]$$

$$\leq K \sum_{k=1}^{\infty} \frac{|\epsilon_k|}{k} [k\mu_k - (k-1)\ \mu_k\ ]$$

$$+ K \sum_n{}'' \frac{|\epsilon_{n+1}|}{\lambda_n (n+1)} [(n+1)\mu_{n+1} - n\mu_n\ ] .$$

Since Σ'' has only the indices n having the property $\lambda_{n+1} > \lambda_n$, it follows that

$$\sum_{n \geq k}{}'' \frac{1}{\lambda_n^2} \leq \sum_{v=\lambda_k}^{\infty} \frac{1}{v^2} \leq K \frac{1}{\lambda_k} ,$$

we have

$$\sum_n{}'' \leq K ,$$

by hypotheses and by (7.6.1) and (7.6.2).

This completes the proof of Theorem 1.

7.7. <u>Proof of Theorems 2 and 3</u>.

For proving Theorems 2 and 3 we use the following lemmas.

<u>Lemma</u> 1.[1] <u>If</u> $\phi_1(t) \in BV(o,\pi)$, <u>then</u>

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1. This is the particular case of Lemma 9 of Prasad and Bhatt [40], when $\alpha = 1$.

$\tau_n(x) = O(1)$, *as* $n \to \infty$.

Lemma 2.[1] *If*

$$\int_0^t u \mid d\, \phi_1(u) \mid = O(t), \quad o \le t \le \pi$$

then $\tau_n(x) = o\ (\log n)$, *as* $n \to \infty$.

We obtain Theorem 2 from Theorem 1 with $\mu_n \equiv 1$ by an appeal to Lemma 1, and we obtain Theorem 3 from Theorem 1, with $\mu_n = \log n$, by an appeal to Lemma 2.

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1. This is the particular case of Lemma 11 of Prasad and Bhatt [40], when $\alpha = 1$.

Chapter 8

ON $|V,\lambda|$– SUMMABILITY FACTORS OF POWER SERIES AND FOURIER SERIES

8.1. If

$$(8.1.1) \qquad \sum_{v=1}^{n} |\,\varkappa_v\,| = O(n) ,$$

as $n \to \infty$, the series $\sum a_n$ is said to be strongly bounded by Cesàro means of order 1, or bounded $[C,1]$. If

$$(8.1.2) \qquad \sum_{v=1}^{n} \frac{|\varkappa_v|}{v} = O(\log n) ,$$

as $n \to \infty$, the series $\sum a_n$ is said to be strongly bounded by ' logarithmic means' with index 1, or bounded $[R,\log n,1]$[1].

Let σ_n and τ_n denote the n^{th} $(C,1)$-means of the sequences $\{s_n\}$ and $\{n\,a_n\}$ respectively, viz.,

$$\sigma_n = \frac{1}{n+1} \sum_{v=0}^{n} s_v ,$$

$$\tau_n = \frac{1}{n+1} \sum_{v=1}^{n} v\,a_v .$$

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1. Pati [34].

Then, since by an identity of Kogbetliantz[1] $n(\sigma_n - \sigma_{n-1}) = \tau_n$, n^{th} total variation of the sequence $\{\sigma_n\}$ is given by

$$(8.1.3)\quad \sum_{v=1}^{n} |\sigma_v - \sigma_{v-1}| = \sum_{v=1}^{n} \frac{|\tau_v|}{v}.$$

The other definitions and notations which are used in this chapter are the same as in the preceeding chapter.

8.2. Let $f(t)$ be a periodic function, with period 2π, integrable in the sense of Lebesgue over $(-\pi,\pi)$, and let

$$(8.2.1)\quad \frac{1}{2}a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$
$$= \frac{1}{2}a_o + \sum_{n=1}^{\infty} A_n(t)$$

denote the Fourier series of the function $f(t)$.

8.3. For any sequence $\{f_n\}$, we write

$$(8.3.1)\quad \Delta^o f_n = f_n,$$
$$\Delta f_n = \Delta^1 f_n = f_n - f_{n+1},$$
$$\Delta^2 f_n = \Delta(\Delta f_n).$$

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1. Kogbetliantz [23].

A sequence $\{f_n\}$ is said to be convex[1], if $\Delta^2 f_n \geq 0$, for $n = 0,1,2,\ldots$

If $\{f_n\}$ is a convex sequence such that the series $\Sigma \frac{f_n}{n}$ is convergent, then it is known that

(i) $\log n \; f_n = o(1)$, as $n \to \infty$ [2]

(ii) $\Sigma \, n \, \Delta^2 f_n < \infty$ [2]

(iii) $\Sigma \, n \, (\log n) \, \Delta^2 f_n < \infty$ [3]

8.4. In the present chapter we raise the question as to what type of sequences of factors $\{\epsilon_n\}$ can be chosen so that the series $\Sigma \, \epsilon_n a_n$ may be summable $|V,\lambda|$, whenever the series $\Sigma \, a_n$ is not summable $|C,1|$, but the total variation of the $|C,1|$- mean of $\Sigma \, a_n$ is of a certain order μ_n, say, when $\{\mu_n\}$ is a positive non-decreasing sequence.

As an answer to this question we establish, in Theorem 1, a result on the $|V,\lambda|$- summability factors for general infinite series, which, in view of (8.3.2), includes the following result as a special case.

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1. Zygmund [50], p. 93.
2. Pati [34].
3. Pati and Sinha [36].

Theorem A.[1] If Σa_n is bounded $[R, \log n, 1]$ and $\{f_n\}$ is a convex sequence such that $\Sigma f_n/n < \infty$ then $\Sigma a_n f_n$ is summable $|C,1|$.

We also establish, as Theorem 2, slightly different type of summability factors theorem for general infinite series, from which we deduce our Theorem 3.

Our main object in the present chapter is to prove, with the help of our Theorems 1 and 2, some general theorems (Theorems 3,4,5 and 6) on the $|V,\lambda|$-summability factors of power series and Fourier series. It is interesting to note that, in view of (8.3.2), our theorems contain, as special cases, the following known results[2] in this line.

Theorem B. If $f(z) = \Sigma c_n z^n$ is a power series of the complex class L , such that

$$\int_o^t |f(e^{i\theta})| \, d\theta = o(|t|) ,$$

as $t \to +o$, and $\{f_n\}$ is a convex sequence such that $\Sigma f_n/n < \infty$, then $\Sigma f_n c_n$ is summable $|C,1|$.[3]

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1. Pati [34], Theorem 1 ; this result has recently been generalized by Ahmad [6] for $|N,p_n|$-summability.
2. These results have recently been generalized by Ahmad [6] for $|N,p_n|$ method.
3. Rajagopal [41], Theorem 1.

Theorem C.[1] If $\{f_n\}$ is a convex sequence such that $\Sigma f_n/n < \infty$, then the series $\Sigma f_n A_n(x)$ is summable $|C,1|$ for almost all values of x.

Theorem D.[2] If $\{f_n\}$ is a convex sequence and the series $\Sigma f_n/\lambda_n$ converges, then the series $\Sigma f_n A_n(x)$ is summable $|V,\lambda|$ almost everywhere.

Theorem E.[3] If $F(x)$ is even, $F(x) \in L^2(-\pi,\pi)$,

$$(8.4.1) \quad \int_0^t |F(x)|^2 dx = O(t),$$

as $t \to +0$, and if $\{f_n\}$ is a convex sequence such that $\Sigma f_n/n < \infty$, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $\Sigma f_n A_n$ is summable $|C,1|$.

Theorem F.[4] If $F(x)$ is even $F(x) \in L(-\pi,\pi)$,

$$(8.4.2) \quad \int_0^t |F(x)|\, dx = O(t),$$

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1. Chow [13].
2. Leindler [27], Theorem 1.
3. Rajgopal [41], Theorem II.
4. Rajgopal [41], Theorem III.

as $t \to +0$, and if $\{f_n\}$ is a convex sequence such that $\Sigma\, n^{-1} f_n < \infty$, then the sequence $\{A_n\}$ of Fourier coefficient of $F(x)$ has the property that $\Sigma (\log \overline{n+1})^{-1/2} f_n A_n$ is summable $|C,1|$.

8.5. We establish the following theorems.

Theorem 1. *If*

$$\sum_{v=1}^{n} \frac{|\tau_v|}{v} = O(\mu_n) ,$$

where $\{\mu_n\}$ is a positive non-decreasing sequence, and if the sequence $\{\epsilon_n\}$ is such that

$$\text{(i)} \qquad \epsilon_n \mu_n = O(1), \; n \Delta \mu_n = O(\mu_n) ,$$

$$\text{(ii)} \qquad \Sigma\, n \mu_n \,|\, \Delta^2 \epsilon_n \,| < \infty ,$$

then the series $\Sigma (n+1)^{-1} \lambda_n \epsilon_n a_n$ *is summable* $|V,\lambda|$.

Theorem 2. *If*

$$\Sigma \,|\, \tau_n | = O(\, n\mu_n \,) ,$$

where $\{\mu_n\}$ is a positive non-decreasing sequence, and if the sequence $\{\epsilon_n\}$ is such that $\Sigma\, \mu_n |\epsilon_n| / \lambda_n < \infty$,

(i) $\epsilon_n \mu_n = O(1)$, $n \Delta \mu_n = O(\mu_n)$,

<u>and</u>

(ii) $\Sigma \mu_n | \Delta^2 \epsilon_n | < \infty$,

<u>then the series</u> $\Sigma \epsilon_n a_n$ <u>is summable</u> $|V,\lambda|$.

<u>Theorem</u> 3. <u>If the sequence</u> $\{\epsilon_n\}$ <u>is such that</u> $\Sigma \frac{|\epsilon_n|}{\lambda_n} < \infty$, <u>and</u>

(i) $\epsilon_n = O(1)$, <u>and</u> (ii) $\Sigma n | \Delta^2 \epsilon_n| < \infty$,

<u>then the series</u> $\Sigma \epsilon_n A_n(x)$ <u>is summable</u> $|V,\lambda|$ <u>for almost all values of</u> x.

<u>Theorem</u> 4. <u>If</u> F(x) <u>is even</u>, $F(x) \in L^2(-\pi,\pi)$,

(8.5.1) $\int_0^t |F(x)|^2 dx = o(t)$,

<u>as</u> $t \to +0$, <u>and if</u> $\{\epsilon_n\}$ <u>satisfies the conditions</u>

(i) $\log n \, \epsilon_n = O(1)$,

<u>and</u>

(ii) $\Sigma n \log n | \Delta^2 \epsilon_n | < \infty$,

<u>then the sequence</u> $\{A_n\}$ <u>of Fourier coefficients of</u> F(x) <u>has the property that</u> $\Sigma(n-1)^{-1} \lambda_n \epsilon_n A_n$ <u>is summable</u> $|V,\lambda|$.

Theorem 5. *If $F(x)$ is even, $F(x) \in L(-\pi,\pi)$,*

$$(8.5.2) \qquad \int_0^t |F(x)| \, dx = O(t) ,$$

as $t \to +0$, and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem 4, then the sequence $\{A_n\}$ of Fourier coefficients of $F(x)$ has the property that $(n+1)^{-1}(\log n)^{-\frac{1}{2}} \lambda_n \epsilon_n A_n$ is summable $|V,\lambda|$.

Theorem 6. *If $f(z) = \Sigma c_n z^n$ is a power series of the complex class L, such that*

$$(8.5.3) \qquad \int_0^t |f(e^{i\theta})| \, d\theta = O(|t|) ,$$

as $t \to +0$, and if $\{\epsilon_n\}$ satisfies the same conditions as in Theorem 4, then $\Sigma(n+1)^{-1} \lambda_n \epsilon_n c_n$ is summable $|V,\lambda|$.

8.6. We use the following Lemmas in the sequel.

Lemma 1.[1] *Let $\{\mu_n\}$ be a positive non-decreasing sequence such that $n \Delta \mu_n = O(\mu_n)$, as $n \to \infty$. If the sequence $\{\epsilon_n\}$ is such that*

---

1. Ahmad [6], Lemma 4.

(i) $\mu_n e_n = O(1)$, *as* $n \to \infty$,

(ii) $\sum n\mu_n \, | \Delta^2 e_n | < \infty$,

then

(a) $\sum \mu_n \, | \Delta e_n| < \infty$,

and

(b) $n\mu_n \, \Delta \mu_n = O(1)$, *as* $n \to \infty$.

Lemma 2. *If* $\bar{e}_n = (n+1)^{-1} \lambda_n e_n$, *then under the hypotheses of Theorem 1*

(a) $\displaystyle\sum_{v=1}^{\infty} \frac{|\bar{e}_v|}{\lambda_n} \, | \tau_v| \leq K$,

(b) $\displaystyle\sum_{v=1}^{\infty} | \Delta e_v| \, | \tau_v | \leq K$,

Proof. (a) We have, as $m \to \infty$,

$$\sum_{v=1}^{m} \frac{|\bar{e}_v| \; |\tau_v|}{\lambda_v} = O\Big(\sum_{v=1}^{m} |e_v| \frac{|\tau_v|}{v} \Big)$$

$$= O\Big(\sum_{v=1}^{m-1} \mu_v |\Delta e_v|\Big) + O(|e_m| \mu_m)$$

$$= O(1),$$

by hypothesis and Lemma 1(a).

(b) Since

$$\Delta \bar{e}_n = \frac{\lambda_n}{n+1} \Delta e_n - \frac{(\lambda_n - \lambda_{n+1})}{n+1} e_{n+1}$$

$$+ \frac{\lambda_{n+1}}{n+v} \frac{e_{n+1}}{n+1} ,$$

we have

$$| \Delta \bar{e}_n | = O(| \Delta e_n | \frac{\lambda_n}{n}) + O(\frac{|e_{n+1}|}{n}) .$$

Hence, as $m \to \infty$,

$$\sum_{v=1}^{m} |\Delta \bar{e}_v| \, |\zeta_v| = O(\sum_{v=1}^{m} \lambda_n |\Delta e_v| \frac{|\zeta_v|}{v})$$

$$+O (\sum_{v=1}^{m} |e_{v+1}| \frac{|\zeta_v|}{v})$$

$$= O(\sum_{v=1}^{m-1} v\mu_v | \Delta^2 e_v |)$$

$$+ O(\sum_{v=1}^{m-1} \mu_v | \Delta e_{v+1} |)$$

$$+ O(m \mu_m | \Delta e_m |)$$

$$+ O(\mu_{m+1} | e_{m+1} |)$$

$$= O(1).$$

by hypotheses and Lemmas 1(a) and 1(b).

This completes the proof of the lemma.

<u>Lemma</u> 3. <u>Under the hypotheses of Theorem</u> 2,

$$\text{(a)} \quad \sum_{v=1}^{\infty} \frac{|e_v|}{\lambda_v} \, |\zeta_v| \leq K ,$$

$$\text{(b)} \quad \sum_{v=1}^{\infty} |\Delta e_v| \, |\zeta_v| \leq K .$$

<u>Proof</u>. (a) We have, as $m \to \infty$

$$\sum_{v=1}^{m} \frac{|e_v|}{\lambda_v} \, |\zeta_v| = O\left(\sum_{v=1}^{m-1} \frac{|e_v|}{\lambda_n \lambda_{n+1}} \, v \, \mu_v \right)$$

$$+ O\left(\sum_{v=1}^{m-1} \frac{|\Delta e_v|}{\lambda_{v+1}} \, v \, \mu_v \right)$$

$$+ O\left(\frac{m \, |e_m|}{\lambda_m} \, \mu_m \right)$$

$$= O\left(\sum_{v=1}^{m-1} \frac{|e_v|}{\lambda_v} \, \mu_v \right)$$

$$+ O\left(\sum_{v=1}^{m-1} \mu_v |\Delta e_v| \right)$$

$$+ O(\mu_m \, e_m) = O(1),$$

by hypotheses.

(b) We have, as $m \to \infty$,

$$\sum_{v=1}^{m} |\Delta e_v| \, |\tau_v|$$

$$= O\left(\sum_{v=1}^{m-1} v \, \mu_v \, | \Delta^2 e_v | \right)$$

$$+ O(m \, \mu_m \, | \Delta e_m |)$$

$$= O(1),$$

by hypotheses and Lemmas 1(a) and 1(b).

This completes the proof of Lemma 3.

Lemma 4.[1] *If* $f(z) = \sum c_n z^n$ *is a power series of complex class* L , *such that*

$$\int_{0}^{t} | f(e^{i\theta}) | \, d\theta = O(|t|),$$

as $t \to +0$, *then* $\sum c_n$ *is bounded* $[F, \log n, 1]$.

Lemma 5.[2] *If* $\sum a_n$ *is bounded* $[F, \log n, 1]$, *then*

$$\sum_{v=1}^{n} \frac{|\tau_v|}{v} = O(\log n),$$

as $n \to \infty$.

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1. Rajgopal [41].
2. Pati [34], p. 294.

Lemma 6.[1] *If* Σa_n *is bounded* $[C,1]$, *then it is* bounded $[R, \log n, 1]$.

Lemma 7.[2] *Let*

$$\tau_n(x) = \frac{1}{n+1} \sum_{v=1}^{n} v\, a_v(x),$$

then

$$\sum_{v=1}^{n} | \tau_v(x) | = o(n) ,$$

for almost all values of x.

Lemma 8.[3] *Let* $F(x)$ *be even*, $F(x) \in L^2(-\pi,\pi)$, *and let* s_n *denote the* n^{th} *partial sum of its Fourier series at the origin. Then, if*

$$\int_0^{\theta} |F(x)|^2\, dx = O(\theta) ,$$

as $\theta \to o$, $\{s_n\}$ *will be summable* $[C,1]$.

Lemma 9.[4] *Let* $f(x)$ *be even*, $f(x) \in L(-\pi,\pi)$, *and*

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1. Ahmad [6], Lemma 8.
2. Chow [14], Lemma 2.
3. Rajgopal [41], Lemma 4.
4. Rajgopal [41].

let s_n *denote the* n^{th} *partial sum of its Fourier series at the origin, then if*

$$\int_0^\theta |f(x)|\, dx = o(\theta),$$

as $\theta \to +0$, *then*

$$\sum_{v=1}^{n} |s_v| = o\left\{n\,(\log n)^{1/2}\right\}.$$

8.6. *Proof of Theorem* 1.

Let $\bar{\epsilon}_n = (n+1)^{-1}\lambda_n \epsilon_n$, and let V_n denote the n^{th} generalized de la Vallée Poussin mean of the series $\sum \bar{\epsilon}_v a_v$. Then, by definition and a well known formula

$$(8.6.1)\qquad \sum_n |V_{n+1} - V_n^*| = \sum_n \left| \frac{1}{\lambda_n\lambda_{n+1}} \sum_{k=n-\lambda_n+1}^{n+1} \left[(\lambda_{n+1}-\lambda_n)(k-n-1) + \lambda_n\right] \bar{\epsilon}_k a_k \right|.$$

Let $\sum_n{}'$ denote the summation over all n satisfying $\lambda_{n+1} = \lambda_n$; and $\sum_n{}''$ the summation over all n where $\lambda_{n+1} > \lambda_n$. Then, Abel's transformation gives that

$$(8.6.2)\qquad \sum_n{}' = \sum_n{}' \frac{1}{\lambda_{n+1}} \left| \sum_{k=n-\lambda_n+2} \frac{\bar{\epsilon}_k}{k}\, k\, a_k \right|$$

$$\leq \sum_n{}' \frac{1}{\lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n} k |\tau_k| \, | \Delta (\frac{\bar{c}_k}{k}) |$$

$$+ \sum_n{}' \frac{1}{\lambda_{n+1}} \frac{\bar{\varepsilon}_{n+1}}{n+1} n \, | \tau_n |$$

$$+ \frac{\varepsilon_{n-\lambda_n+2}}{n-\lambda_n+2} (n-\lambda_n+2) \, | \tau_{n-\lambda_n+1} |$$

$$= \sum_{n1}{}' + \sum_{n2}{}' + \sum_{n3}{}' \text{, say.}$$

Since the inside lower insides $n-\lambda_n+2$ in $\sum_{n1}{}'$ are strictly increasing, we have

$$(8.6.3) \qquad \sum_{n1}{}' \leq K \sum_n{}' \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+2}^{n} k \, |\tau_k| \, |\Delta(\frac{\bar{\varepsilon}_k}{k}) |$$

$$= K \sum_{k=1}^{\infty} k \, |\tau_k| \, | \Delta (\frac{\bar{\varepsilon}_k}{k}) | \sum_{n=k}^{k+\lambda_k-1} \frac{1}{\lambda_n}$$

$$\leq K \sum_{k=1}^{\infty} k \, | \tau_k | \, | \Delta (\frac{\bar{\varepsilon}_k}{k}) |$$

$$\leq K \sum_{k=1}^{\infty} \frac{|\tau_k|}{k} \, | \varepsilon_k | + K \sum_{k=1}^{\infty} | \tau_k | \, | \Delta \bar{\varepsilon}_k |$$

$$\leq K \sum_{k=1}^{\infty} \frac{| \tau_k | \, | \bar{\varepsilon}_k |}{\lambda_k} + K \sum_{k=1}^{\infty} | \tau_k | \, | \Delta \bar{\varepsilon}_k |$$

$$\leq K \, ,$$

by hypotheses and Lemma 2 ; and

$$(6.6.4) \qquad \Sigma'_{n2} + \Sigma'_{n3} = K \sum_{n=1}^{\infty} \frac{|\bar{\epsilon}_n|}{\lambda_n} \, |\tau_n| \le K ,$$

by hypotheses and Lemma 2(a).

The estimate of Σ''_n is some what tricky. We obtain, with the help of Abel's transformation, that

$$(6.6.5) \qquad \Sigma''_n = \Sigma''_n \frac{1}{\lambda_n \lambda_{n+1}} \left| \sum_{k=n-\lambda_n+1}^{n+1} (\lambda_n - n - 1 + k) \frac{\bar{\epsilon}_k}{k} \, k \, a_k \right|$$

$$\le K \, \Sigma''_n \frac{1}{\lambda_n^2} \sum_{k=n-\lambda_n+1}^{n} k \, |\tau_k| \, \times$$

$$\times \left| \Delta \left[(\lambda_n - n - 1 + k) \frac{\bar{\epsilon}_k}{k} \right] \right|$$

$$+ \Sigma''_n \frac{(n-\lambda_n+1)}{\lambda_n \lambda_{n+1}} |\tau_{n-\lambda_n+1}| \frac{|\bar{\epsilon}_{n-\lambda_n+2}|}{n-\lambda_n+2}$$

$$+ \Sigma''_n \frac{(n+1)}{\lambda_n \lambda_{n+1}} |\tau_{n+1}| \, \lambda_n \frac{|\bar{\epsilon}_{n+1}|}{n+1}$$

$$= \Sigma''_{n1} + \Sigma''_{n2} + \Sigma''_{n3}, \text{ say .}$$

Since

$$\left| \Delta \left[(\lambda_n - n - 1 + k) \frac{\bar{\epsilon}_k}{k} \right] \right| \le \lambda_k \left| \Delta \left(\frac{\bar{\epsilon}_k}{k} \right) \right| + \frac{|\bar{\epsilon}_k|}{k}$$

$$\leq \frac{\lambda_k}{k+1} |\Delta \bar{\epsilon}_k| + \lambda_k \frac{|\bar{\epsilon}_k|}{k(k+1)} + \frac{|\bar{\epsilon}_k|}{k} ,$$

we have

$$\Sigma_{n^1}'' \leq \sum_{k=2}^{\infty} \lambda_k [\, |\zeta_k||\Delta \bar{\epsilon}_k| + |\zeta_k| \frac{|\bar{\epsilon}_k|}{k} + |\zeta_k||\bar{\epsilon}_k| \,] \times$$

$$\times \sum_{n\geq k}'' \frac{1}{\lambda_n^2}$$

$$\leq K \sum_{k=2}^{\infty} |\zeta_k||\Delta \bar{\epsilon}_k| + K \sum_{k=2}^{\infty} \frac{|\zeta_k|}{\lambda_k} |\bar{\epsilon}_k|$$

$$+ K \sum_{k=2}^{\infty} \frac{|\zeta_k|}{\lambda_k} |\bar{\epsilon}_k|$$

$$\leq K ,$$

by hypotheses and Lemmas 2(a) and 2(b) and because Σ'' have only the indices n having the property that $\lambda_{n+1} > \lambda_n$,

$$\sum_{n\geq k}'' \frac{1}{\lambda_n^2} \leq \sum_{v=\lambda_k}^{\infty} \frac{1}{v^2} \leq K \frac{1}{\lambda_k} .$$

Also $\quad \Sigma_{n^2}'' + \Sigma_{n^3}'' \leq \sum_{n=1}^{\infty} |\zeta_{n+1}| \frac{|\bar{\epsilon}_{n+1}|}{\lambda_{n+1}} \leq K ,$

by hypotheses and Lemma 2(a).

This terminates the proof of Theorem 1.

8.7. Proof of Theorem 2.

Here we derive the proof from the proof of Theorem 1. We use V_n^* to denote the n^{th} generalized de la Vallée mean of the series $\Sigma\, \varepsilon_n a_n$ instead of the series $\Sigma\, \bar{\varepsilon}_n\, \bar{a}_n$.

Thus, in order to prove the theorem we have to show that

$$\Sigma_n' \leq \Sigma_{n1}' + \Sigma_{n2}' + \Sigma_{n3}' \leq K,$$

and

$$\Sigma_n'' \leq \Sigma_{n1}'' + \Sigma_{n2}'' + \Sigma_{n3}'' \leq K,$$

where the Σ_n''s, and Σ_n'''s, are the same as in the proof of Theorem 1.

Now, from (8.6.2) and (8.6.3)

$$\Sigma_{n1}' \leq K \sum_{k=1}^{\infty} \frac{|\zeta_k|}{\lambda_k} |\varepsilon_k| + K \sum_{k=1}^{\infty} |\zeta_k|\,|\Delta \varepsilon_k|$$

$$\leq K,$$

by hypotheses and Lemma 3, and

$$\Sigma_{n2}' + \Sigma_{n3}' = K \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{\lambda_n} |\zeta_n| \leq K,$$

by Lemma 3(a).

Similarly,

$$\Sigma_{n1}'' \leq K \sum_{k=2}^{\infty} |\zeta_k|\,|\Delta \varepsilon_k| + K \sum_{k=2}^{\infty} \frac{|\zeta_k|}{\lambda_k} |\varepsilon_k| \leq K,$$

by Lemma 3

and

$$\Sigma_1'' + \Sigma_2'' \leq K \sum_{n=1}^{\infty} \frac{|\zeta_{n+1}|}{\lambda_{n+1}} |\epsilon_{n+1}| \leq K ,$$

by hypotheses and Lemmas 3(e).

This terminates the proof of Theorem 2.

8.8. Proofs of Theorem 3,4,5 and 6.

Theorem 3 is obtained from Theorem 2 by putting $\lambda_n \equiv 1$, and by an appeal to Lemma 7.

Theorem 4 can easily be obtained from Theorem 1, with $\mu_n = \log n$, by successive applications of Lemmas 8 and 6 and 5.

[illegible] Theorem 5 from Theorem 1 with $\mu_n = (\log n)^{3/2}$, and with $\epsilon_n/(\log n)^{1/2}$ in place of ϵ_n, by an appeal to Lemma 9 and by using the fact that

$$\sum_{v=1}^{n} |\alpha_v| = O\left\{n (\log n)^{1/2}\right\}$$

implies

$$\sum_{v=1}^{n} \frac{|\zeta_v|}{v} = O\left\{(\log n)^{3/2}\right\}.$$

Finally, we obtain Theorem 6 from Theorem 1, with $\mu_n = \log n$, by appealing to Lemmas 4 and 5.

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